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RADC-TR-81-380
Phase Report
January 1982

ELECTROMAGNETIC COUPLING THROUGH APERTURES

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RADC-TR-81-380 has been reviewed and is approved for publication.

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER RADC-TR-81-380	2. GOVT ACCESSION NO. AD-A112630	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ELECTROMAGNETIC COUPLING THROUGH APERTURES		5. TYPE OF REPORT & PERIOD COVERED Phase Report
		6. PERFORMING ORG. REPORT NUMBER N/A
7. AUTHOR(s) Joseph R. Mautz Roger F. Harrington David T. Auckland		8. CONTRACT OR GRANT NUMBER(s) F30602-79-C-0011
9. PERFORMING ORGANIZATION NAME AND ADDRESS Syracuse University Department of Electrical And Computer Engrg. Syracuse NY 13210		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62702F 23380317
11. CONTROLLING OFFICE NAME AND ADDRESS Rome Air Development Center (RBCT) Griffiss AFB NY 13441		12. REPORT DATE January 1982
		13. NUMBER OF PAGES 128
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Same		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Same

18. SUPPLEMENTARY NOTES

RADC Project Engineer: Roy F. Stratton (RBCT)

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Aperture Admittance	Generalized Network Parameters
Aperture and Conducting Body	Method of Moments
Aperture in a Screen	Narrow Slots
Cavity-backed Aperture	Waveguide-fed Apertures

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report is a tutorial presentation on the use of the generalized network approach to aperture problems. The general theory is given and applied to the cases of electromagnetic transmission through an aperture in a conducting plane, a waveguide-fed aperture, a cavity-backed aperture, a narrow slot, and an aperture backed by a conducting body. The last two problems demonstrate the phenomena of aperture resonance for slots in thick conductors and apertures backed by conducting bodies. At

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resonance, an aperture can transmit orders of magnitude greater power than the same aperture not resonated. Such behavior should be taken into account in the engineering analysis of electromagnetic interference and compatibility problems involving apertures.

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CONTENTS

Chapter I

BASIC CONCEPTS

1-1   INTRODUCTION

The coupling of electromagnetic fields between isolated regions

through one or more apertures is a widely encountered problem in elec-

tromagnetics.   Some examples are cavity-to-cavity coupling, waveguide-

to-waveguide coupling, waveguide-to-exterior space coupling, and so on.

The general problem consists of two or more regions of space coupled by

one or more apertures.   There can be sources in one or more regions,

and material bodies in one or more regions.   Figure 1-1 shows a typical

problem of two regions coupled by an aperture.   For this example, it

is assumed that sources exist in region a, and a material body exists

in region b.   The boundary of the regions is considered to be a perfect
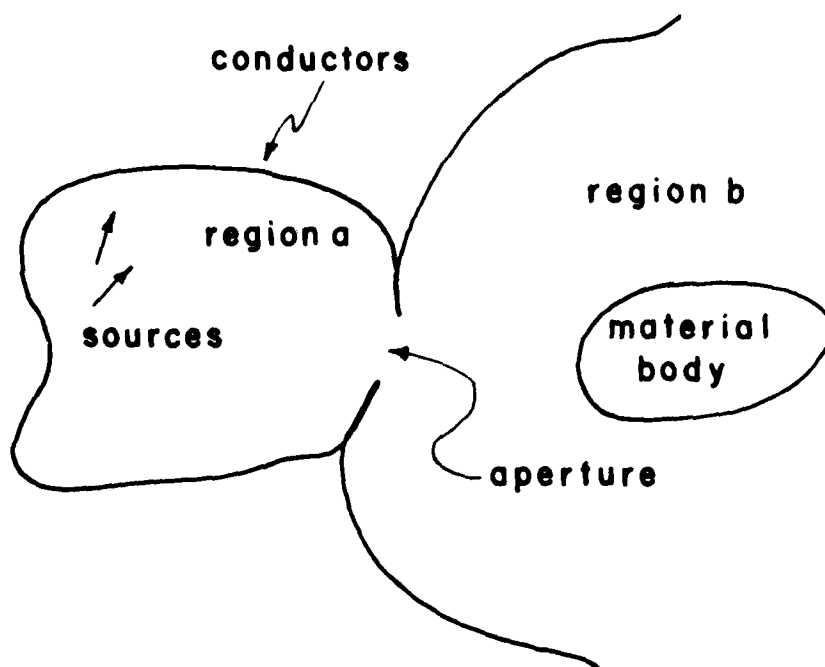
Fig. 1-1.   A typical aperture coupling problem.

electric conductor, but other types of impermeable boundaries may be used. Each region may be closed, as region a is shown in Fig. 1-1, or may be open (extend to infinity), as region b is shown.

The present chapter is intended to serve as a short summary of the principal relationships and notation that we use throughout the report. For some readers this should be only a quick review. Others, less familiar with electromagnetic theory, should refer to some of the well-known textbooks [1] to [7]. (Bracketed numbers denote references which are listed at the end of each chapter.)

1-2   FIELDS AND SOURCES

For our purposes, the electromagnetic field is expressed in terms of two vectors:

$\underline{E}$ - the electric field

$\underline{H}$ - the magnetic field

The source of the field may be of two types

$\underline{J}$ - the electric current

$\underline{M}$ - the magnetic current

We use these symbols for both volume densities and for surface densities. It should be clear from the text or equations which density is meant. The medium in which the field exists is characterized by three constitutive parameters.

ε - the permittivity or capacitivity

μ - the permeability or inductivity

σ - the conductivity

ε and μ may be complex to allow for energy dissipation, in which case σ need not be shown explicitly.

In any region of space for which $\underline{E}$ and $\underline{H}$ are differentiable, the field satisfies the Maxwell curl equations

$$\underline{\nabla} \times \underline{H} = j\omega\varepsilon \, \underline{E} + \underline{J}$$

$$-\underline{\nabla} \times \underline{E} = j\omega\mu \, \underline{H} + \underline{M} \tag{1-1}$$

where $\underline{\nabla} \times$ is the curl operator and ω is the angular frequency. The divergence of (1-1) yields the Maxwell divergence equations

$$\underline{\nabla} \cdot \varepsilon \underline{E} = q$$

$$\underline{\nabla} \cdot \mu \underline{H} = m \tag{1-2}$$

where $\underline{\nabla} \cdot$ is the divergence operator and the two scalars are

        q – the electric charge density

        m – the magnetic charge density

These are related to the electric and magnetic currents by the equations of continuity

$$\underline{\nabla} \cdot \underline{J} = - \, j\omega q$$

$$\underline{\nabla} \cdot \underline{M} = - \, j\omega m \tag{1-3}$$

Here the symbols q and m denote volume densities, but we use the same symbols for surface densities. It should be clear from the text and equations which density is meant.

In any region that is source-free $(\underline{J} = \underline{M} = \underline{0})$, linear, and homogeneous (ε and μ independent of position), the curl of equations (1-1) leads to the vector Helmholtz equations

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$$-\underline{\nabla} \times \underline{\nabla} \times \underline{E} + k^2\underline{E} = \underline{0}$$

$$-\underline{\nabla} \times \underline{\nabla} \times \underline{H} + k^2\underline{H} = \underline{0}$$

(1-4)

where $k = \omega\sqrt{\varepsilon\mu}$ is the wavenumber. The rectangular components of $\underline{E}$ and $\underline{H}$ satisfy the scalar Helmholtz equation

$$\nabla^2 E_i + k^2 E_i = 0$$

$$\nabla^2 H_i + k^2 H_i = 0$$

(1-5)

where $i = x$, y, or z. In addition to (1-5), the vectors must satisfy $\underline{\nabla} \cdot \underline{E} = 0$ and $\underline{\nabla} \cdot \underline{H} = 0$.

1-3 POTENTIALS

It is often convenient to express the electromagnetic fields in terms of auxiliary functions, called potentials. In a homogeneous region we have the representation

$$\underline{E} = -j\omega\underline{A} - \underline{\nabla}\phi$$

$$\underline{H} = \frac{1}{\mu} \underline{\nabla} \times \underline{A}$$

(1-6)

where the potentials are

$\underline{A}$ – the magnetic vector potential

ϕ – the electric scalar potential

The potentials A and ϕ satisfy the Lorentz gauge

$$\underline{\nabla} \cdot \underline{A} = -j\omega\mu\varepsilon\phi$$

(1-7)

which is analogous to the equation of continuity (1-3). Potentials with other gauges are sometimes useful, but we do not consider them in this report.

Alternatively, the field in a homogeneous region can be represented as

$$\underline{E} = - \frac{1}{\epsilon} \underline{\nabla} \times \underline{F}$$

$$\underline{H} = - j\omega\underline{F} - \underline{\nabla}\psi$$

(1-8)

where the potentials are

$\underline{F}$ - the electric vector potential

ψ - the magnetic scalar potential

The potentials $\underline{F}$ and ψ also satisfy the Lorentz gauge

$$\underline{\nabla} \cdot \underline{F} = - j\omega\mu\epsilon\psi$$

(1-9)

The potentials $\underline{F}$ and ψ are said to be dual to the potentials $\underline{A}$ and ϕ.

In general, the field in a homogeneous region can be represented as a superposition of (1-6) and (1-8), or

$$\underline{E} = - \frac{1}{\epsilon} \underline{\nabla} \times \underline{F} - j\omega\underline{A} - \underline{\nabla}\phi$$

$$\underline{H} = \frac{1}{\mu} \underline{\nabla} \times \underline{A} - j\omega\underline{F} - \underline{\nabla}\psi$$

(1-10)

In a source-free homogeneous region, the rectangular components of $\underline{A}$ and $\underline{F}$ and the scalars ϕ and ψ satisfy the Helmholtz equation

$$\nabla^2\pi + k^2\pi = 0$$

(1-11)

When one solution of (1-11) is taken for a rectangular component of $\underline{A}$, and another solution of (1-11) is taken for a rectangular component of $\underline{F}$, the potentials are sometimes called Hertzian potentials.

One advantage of using potentials is that the conditions $\underline{\nabla} \cdot \underline{E} = 0$ and $\underline{\nabla} \cdot \underline{H} = 0$ are automatically satisfied. Another advantage is that the potentials can be simply related to the sources, as we show in the next section.

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1-4 POTENTIAL INTEGRALS

Given an electric current $\underline{J}$ and its associated electric change q
in an infinite homogeneous medium characterized by μ and ε, the potential
integrals are

$$\underline{A}(\underline{r}) = \mu \iiint \underline{J}(\underline{r}') \ G(\underline{r},\underline{r}')d\tau'$$

$$(1\text{-}12)$$

$$\phi(\underline{r}) = \frac{1}{\varepsilon} \iiint q(\underline{r}') \ G(\underline{r},\underline{r}')d\tau'$$

where the free space Green's function G is

$$G(\underline{r},\underline{r}') = \frac{e^{-jk|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|} \tag{1-13}$$

In (1-12) and (1-13), $\underline{r}$ is the radius vector to the field point and $\underline{r}'$
is the radius vector to the source point. The sources $\underline{J}$ and q are con-
sidered to be volume densities in (1-12), as evidenced by the triple
integral sign with $d\tau'$ representing a volume element of integration.
If the current and charge are surface densities, it is merely necessary
to change to a surface integration, and similarly for line densities.
The electromagnetic field associated with $\underline{A}$ and ϕ is given by (1-6).
The charge q is related to the current $\underline{J}$ by the equation of continuity (1-3).

For the dual case of a magnetic current $\underline{M}$ and its associated mag-
netic charge m in an infinite homogeneous medium characterized by ε and μ,
the potential integrals are

$$\underline{F}(\underline{r}) = \varepsilon \iiint \underline{M}(\underline{r}') \ G(\underline{r},\underline{r}')d\tau'$$

$$(1\text{-}14)$$

$$\psi(\underline{r}) = \frac{1}{\mu} \iiint m(\underline{r}') \ G(\underline{r},\underline{r}')d\tau'$$

where the Green's function G is again given by (1-13). The electromagnetic field associated with $\underline{F}$ and ψ is given by (1-8). The charge m is related to the current $\underline{M}$ by the equation of continuity (1-3).

If the problem of interest is two-dimensional, that is, if the sources are infinite in extent in one rectangular direction and independent of that direction, the Green's function must be changed to

$$G(\underline{r}, \ \underline{r}') = \frac{1}{4j} \, H_0^{(2)}(k|\underline{r} - \underline{r}'|) \qquad (1-15)$$

Here $H_0^{(2)}$ is the zero order Hankel function of the second kind, equal to $J_0 - jN_0$, where J_0 is the zero order Bessel function of the first kind and N_0 is the zero order Bessel function of the second kind (or Neumann function). The potential integrals (1-12) and (1-14) must now be surface integrals in the cross sectional surface transverse to the direction of invariance of the current.

1-5 CONDITIONS AT SURFACES

There may be discontinuities in the electromagnetic field at sheets of current and at surface discontinuities of the constitutive parameters. In Fig. 1-2, let S represent a surface between two regions, a and b, and let $\underline{n}$ be the unit vector normal to S pointing into region a. There may be an electric surface current $\underline{J}$ and/or a magnetic surface current $\underline{M}$ on S. (In figures, we use a single headed arrow to denote electric current and a double headed arrow to denote magnetic current.) The tangential components of the field on S then obey the conditions

$$\left. \begin{array}{l} \underline{J} = \underline{n} \times (\underline{H}^a - \underline{H}^b) \\[2ex] \underline{M} = (\underline{E}^a - \underline{E}^b) \times \underline{n} \end{array} \right\} \quad \text{on S} \qquad (1-16)$$

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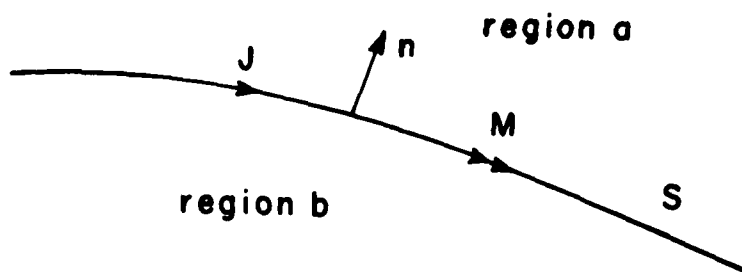
Fig. 1-2. A boundary surface S between regions a and b, possibly
supporting surface currents $\underline{J}$ and $\underline{M}$.

where superscript a on a vector denotes that it is evaluated in or is
peculiar to region a, and similarly for the superscript b. The field
vectors in (1-16) are evaluated on the surface S, as implied by "on S"
in (1-16), with the superscript denoting on which side of S.

The normal components of the fields on S obey the conditions

$$\left.\begin{array}{l} q = \underline{n} \cdot (\varepsilon^a \underline{E}^a - \varepsilon^b \underline{E}^b) \\[2em] m = \underline{n} \cdot (\mu^a \underline{H}^a - \mu^b \underline{H}^b) \end{array}\right\} \text{ on S} \qquad (1\text{-}17)$$

where q and m are the surface densities of electric and magnetic charges on
S, ε^a, μ^a are ε, μ in region a, and ε^b, μ^b are ε, μ in region b. The q
and m are related to $\underline{J}$ and $\underline{M}$ by the equation of continuity (1-3), except
that now $\underline{\nabla} \cdot$ represents the surface divergence operator.

If there is no surface current on S, then the left-hand sides of
(1-16) are zero and

$$\left.\begin{array}{l} \underline{n} \times (\underline{H}^a - \underline{H}^b) = \underline{0} \\[2em] \underline{n} \times (\underline{E}^a - \underline{E}^b) = \underline{0} \end{array}\right\} \text{ on S} \qquad (1\text{-}18)$$

These equations state that the tangential components of $\underline{E}$ and $\underline{H}$ must be continuous across S. There are no surface densities of induced current at a boundary between two media (perfect conductors excepted). Hence, the tangential components of $\underline{E}$ and $\underline{H}$ must be continuous across the boundary between two media (perfect conductors excepted).

Similarly, if there is no surface charge on S, the left-hand sides of (1-17) are zero and

$$\left. \begin{array}{l} \underline{n} \cdot \varepsilon^a \underline{E}^a = \underline{n} \cdot \varepsilon^b \underline{E}^b \\[2ex] \underline{n} \cdot \mu^a \underline{H}^a = \underline{n} \cdot \mu^b \underline{H}^b \end{array} \right\} \quad \text{on S} \qquad (1\text{-}19)$$

These equations state that the normal components of $\varepsilon\underline{E}$ and $\mu\underline{H}$ must be continuous across S. In the time-harmonic case, there are no surface densities of induced charge at a boundary between two media (perfect conductors excepted). Hence, the normal components of $\varepsilon\underline{E}$ and $\mu\underline{H}$ must be continuous across the boundary between two media (perfect conductors excepted).

In the special case of a perfect electric conductor, no field exists internal to the perfect conductor, and a surface current $\underline{J}$ exists on the surface. If region b is a perfect conductor, boundary conditons (1-16) become

$$\left. \begin{array}{l} \underline{J} = \underline{n} \times \underline{H}^a \\[2ex] \underline{0} = \underline{n} \times \underline{E}^a \end{array} \right\} \quad \text{on S} \qquad (1\text{-}20)$$

Hence, the tangential component of $\underline{E}$ is zero at the surface of a perfect electric conductor, and the tangential component of $\underline{H}$ is equal to $\underline{J}$ rotated 90°. Also, (1-17) becomes the boundary conditions

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$$q = \underline{n} \cdot \varepsilon^a \underline{E}^a \left.\right\}$$

$$0 = \underline{n} \cdot \mu^a \underline{H}^a \left.\right\}$$ on S   (1-21)

Hence, the normal component of $\mu\underline{H}$ is zero at the surface of a perfect electric conductor, and the normal component of $\varepsilon\underline{E}$ is equal to the surface charge density.

1-6 THE EQUIVALENCE PRINCIPLE

Many source distributions outside a given region can produce the same field within the region. Two sources producing the same field in a region are said to be equivalent within that region. When we are interested in the field in a given region, we do not need to know the actual sources. Equivalent sources serve as well.

A detailed discussion of the equivalence principle can be found in [1]. Basically, it involves dividing space into two (or more) regions and assuming Maxwellian fields in each region. There is generally a discontinuity in the field at the surface S which forms the common boundary between the two regions. On this boundary we assume surface currents $\underline{J}$ and $\underline{M}$ which are related to the fields by (1-16). We then have a Maxwellian field everywhere and the sources which support it. If there is a one-to-one correspondence between a field and its sources, we have then found equivalent sources for the field in a given region. This one-to-one correspondence is always obtained in the case of lossy media. It is not always obtained in the loss-free case, as noted in Section 3-3 of [1].

The equivalence principle is best illustrated by means of a fairly general example. Consider Fig. 1-3 and let S represent a mathematical
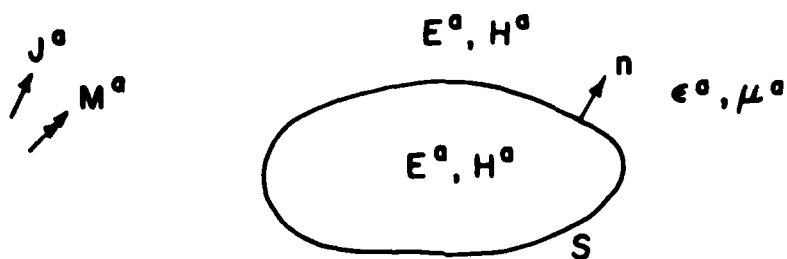
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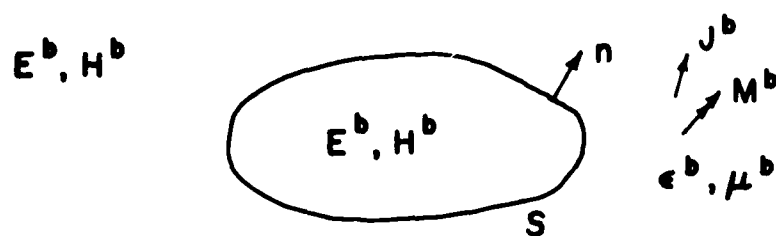
(b) original b problem
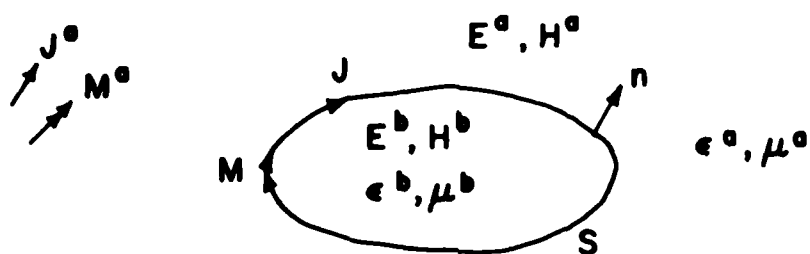
(c) equivalent to a external to S
and to b internal to S

Fig. 1-3.   Illustration of the Equivalence Principle.

surface defined in space. In Fig. 1-3a, let $\underline{J}^a$, $\underline{M}^a$ be impressed sources producing a field $\underline{E}^a$, $\underline{H}^a$ everywhere in a medium characterized by constitutive parameters μ^a and ε^a. In Fig. 1-3b, let $\underline{J}^b$, $\underline{M}^b$ be impressed sources producing a field $\underline{E}^b$, $\underline{H}^b$ everywhere in a medium characterized by constitutive paramete.. μ^b and ε^b. We now construct the equivalent problem of Fig. 3c as follows. External to S, we specify that the sources are $\underline{J}^a$, $\underline{M}^a$, the field is $\underline{E}^a$, $\underline{H}^a$, and the medium is μ^a, ε^a. Internal to S, we specify that the field is $\underline{E}^b$, $\underline{H}^b$ and the medium is μ^b, ε^b. To support the discontinuity in fields on S, we must satisfy (1-16), or

$$\left.\begin{array}{l} \underline{J} = \underline{n} \times (\underline{H}^a - \underline{H}^b) \\[2mm] \underline{M} = (\underline{E}^a - \underline{E}^b) \times \underline{n} \end{array}\right\} \text{ on S} \qquad (1\text{-}22)$$

These are called the equivalent currents on S. Assuming that the relationship between the field and its sources is one-to-one, we know that the equivalent sources in Fig. 1-3c must produce the postulated field.

Some important special cases are: (1) the field in one region is assumed to be the null field, (2) the medium in one region is assumed to be a perfect electric conductor, and (3) the medium in one region is assumed to be a perfect magnetic conductor. These cases are discussed in detail in Section 3-5 of [1], and we use such cases in the next chapter.

1-7 INTEGRAL EQUATIONS

For some geometrically simple problems, solutions for the field can be obtained by solving the differential equations. For more complicated geometries, it becomes more convenient to use integral equations or, more generally to use integro-differential equations to obtain

solutions. These solutions are usually approximate, and one method for obtaining them is the method of moments, discussed in the next section.

A general procedure for establishing integral equations is to use the potential integrals to find the fields in terms of the sources (actual or equivalent) and then to relate the fields to the sources by boundary conditions or constitutive relationships. The procedure is best illustrated by an example.

Let Fig. 1-4 represent a dielectric body V in free space, excited by an impressed field $\underline{E}^i$, which can be thought of as the field which exists due to external sources when the body is absent. Let the constitutive parameters of free space be denoted by ε_o, μ_o and those of the dielectric body by ε, μ_o. The total field within V induces a polarization current $\underline{J}$ related to the total field $\underline{E}$ by

$$\underline{J} = j\omega\ (\varepsilon - \varepsilon_o)\underline{E} \qquad \text{in V} \qquad (1-23)$$

In other words, $\underline{J}$ is the excess displacement current over what would

Fig. 1-4. A dielectric body in free space excited by an impressed field $\underline{E}^i$.

exist there if V contained free space, $\underline{E}$ being unchanged. The total

field $\underline{E}$ is the sum of the impressed field $\underline{E}^i$ plus the scattered (or

secondary) field $\underline{E}^s$ produced by $\underline{J}$, that is

$$\underline{E} = \underline{E}^i + \underline{E}^s(\underline{J}) \tag{1-24}$$

We can calculate $\underline{E}^s(\underline{J})$ by the potential integrals, section 1-4, but

choose not to write down the explicit formulas. For now, it is suf-

ficient to note that $\underline{E}^s$ is linearly related to $\underline{J}$, that is, $\underline{E}^s(\underline{J})$ is a

linear operator. It involves an integral and some derivatives and

should properly be called an integro-differential operator. For bre-

vity, we shall call it simply an integral operator.

We now use the constitutive relationship (1-23) in (1-24) to

obtain

$$\frac{\underline{J}}{j\omega(\varepsilon - \varepsilon_o)} = \underline{E}^i + \underline{E}^s(\underline{J}) \quad \text{in } V \tag{1-25}$$

This can be rearranged into the form

$$\underline{L}(\underline{J}) = \underline{E}^i \tag{1-26}$$

in V, where

$$\underline{L}(\underline{J}) = \frac{\underline{J}}{j\omega(\varepsilon - \varepsilon_o)} - \underline{E}^s(\underline{J}) \tag{1-27}$$

is a linear operation. In (1-26), the impressed (or incident) field $\underline{E}^i$

is known, $\underline{L}$ is a known integral operator, and $\underline{J}$ is the unknown to be

found.

1-8 METHOD OF MOMENTS

The method of moments is a general procedure for solving operator equations of the form (1-26). Its mathematical foundations lie in the theory of projections in linear inner product spaces. A detailed but elementary exposition of the procedure can be found in reference [8]. The following is a short outline of the method.

Suppose we have an operator equation of the form

$$L(f) = g \qquad (1-28)$$

where L is a linear operator, g is a known function, and f is an unknown function to be determined. We must define an inner product $<f, g>$ (or symmetric product if f and g are complex) such that

(a) $<f, g> = <g, f>$

(b) $<\alpha f_1 + \beta f_2, g> = \alpha<f_1, g> + \beta<f_2, g>$ \qquad (1-29)

(c) $<f, f^*> > 0$ \qquad if $f \neq 0$

(d) $<f, f^*> = 0$ \qquad if and only if $f = 0$

Here α and β are scalars, and * denotes complex conjugate. In a complex Hilbert space the usual inner product is $<f, g^*>$, and the conjugate operation is not shown explicitly. In electromagnetic theory it is more convenient to use the symmetric product $<f, g>$ and show the conjugation explicitly when needed.

We next choose a set of linearly independent expansion functions or basis functions $\{f_1, f_2, \ldots\}$ in the domain of L and represent f as the linear combination

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$$f \approx \sum_n \alpha_n f_n \qquad (1-30)$$

where the α_n are scalars to be determined. For computational purposes the representation is usually approximate, as shown. Substituting (1-30) into (1-28) and using the linearity of L, we obtain

$$\sum_n \alpha_n L\, f_n \approx g \qquad (1-31)$$

We next define a set of linearly independent testing functions or weighting functions $\{w_1, w_2, \ldots\}$ in the range of L and take the inner product (or symmetric product) of (1-31) with each w_m. The inner product is linear, and the result is

$$\sum_n \alpha_n \langle w_m, Lf_n \rangle = \langle w_m, g \rangle \qquad (1-32)$$

$m = 1, 2, 3, \ldots$. This is a set of linear equations which can be written in matrix form as

$$[\ell]\vec{\alpha} = \vec{g} \qquad (1-33)$$

where

$$[\ell] = \begin{bmatrix} \langle w_1, Lf_1 \rangle & \langle w_1, Lf_2 \rangle \cdots \\ \langle w_2, Lf_1 \rangle & \langle w_2, Lf_2 \rangle \cdots \\ \cdots \cdots \cdots \cdots \cdots \cdots \end{bmatrix} \qquad (1-34)$$

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \vdots \end{bmatrix} \qquad \vec{g} = \begin{bmatrix} \langle w_1, g \rangle \\ \langle w_2, g \rangle \\ \vdots \\ \vdots \end{bmatrix} \qquad (1-35)$$

If the number of expansion functions equals the number of testing functions,

$[\ell]$ will be a square matrix. The $\vec{\alpha}$ and $\vec{g}$ are column matrices, also called column vectors. If $[\ell]$ is nonsingular, its inverse $[\ell]^{-1}$ exists, and the $\vec{\alpha}$ is then given by

$$\vec{\alpha} = [\ell]^{-1}\vec{g} \tag{1-36}$$

The solution for f is then given by (1-30) where α_n are the components of $\vec{\alpha}$ obtained from (1-36).

By defining a row matrix of the expansion functions as (where the tilde is used to denote the transpose of a matrix in general)

$$\tilde{f} = [f_1 \quad f_2 \quad f_3 \cdot \cdot \cdot] \tag{1-37}$$

we can write the solution concisely in the form

$$f \approx \tilde{f}\,\vec{\alpha} = \tilde{f}[\ell]^{-1}\vec{g} \tag{1-38}$$

Finally, instead of the function f itself, we are often interested in some functional (number) ρ which depends linearly on f. This can be expressed in the general form

$$\rho = <h,\ f> \tag{1-39}$$

where h is a known function (determined as a part of formulating the problem). Substituting (1-38) into (1-39), we have

$$\rho \approx \tilde{h}[\ell]^{-1}\vec{g} \tag{1-40}$$

where $\tilde{h}$ is the row vector

$$\tilde{h} = [<f_1,\ h> \quad <f_2,\ h> \cdot \cdot \cdot] \tag{1-41}$$

In general, we call $\tilde{h}$ a measurement vector. (It is characteristic of the particular measurement ρ we wish to perform.) Every linear measurement (functional) can be expressed as a matrix contraction of the form (1-40).

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1-9 REFERENCES FOR CHAPTER I

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Chapter II

A GENERALIZED NETWORK FORMULATION FOR APERTURE PROBLEMS

2-1 INTRODUCTION

This chapter considers a general formulation for a two-region aperture problem. First an operator equation is obtained in terms of an unknown equivalent magnetic current, and this is then reduced to a matrix equation via the method of moments. The only coupling is through the aperture, whose characteristics are expressed by aperture admittance matrices, one for each region. These admittance matrices depend only on the region being considered, being independent of the other region. The aperture coupling is then expressible as the sum of the two independent aperture admittance matrices, with source terms related to the incident magnetic field. This result can be interpreted in terms of generalized networks as two N-port networks connected in parallel with current sources. The resultant solution is equivalent to an N-term variational solution.

Since the problem is divided into two mutually exclusive parts, one can separately solve a few canonical problems, such as apertures in conducting screens, in waveguides, and in cavities, and then combine them in various permutations. Computer programs can be developed to calculate the aperture admittance matrices for classes of canonical problems, such as apertures of arbitrary shape in conducting planes, in square waveguides, and in rectangular cavities. Such programs can then serve as broad and versatile tools for designing electromagnetic networks with aperture coupling.

2-2 GENERAL FORMULATION

Figure 2-1 represents the general problem of aperture coupling between two regions, called region a and region b. In region a there are impressed sources $\underline{J}^i$, $\underline{M}^i$, and region b is assumed source free. The more general case of sources in both region a and region b can be treated as the superposition of two problems, one with sources in region a only, plus one with sources in region b only. Each region of Fig. 2-1 is shown to be bounded by an electric conductor, although other types of electromagnetic isolation may be used. Region a is shown closed and region b is shown open to infinity, although each region may be open or closed. The equivalence principle, discussed in Sec. 1-6, is used to divide the problem into two equivalent

CONDUCTOR

REGION a

$\uparrow M^i$

$\uparrow J^i$
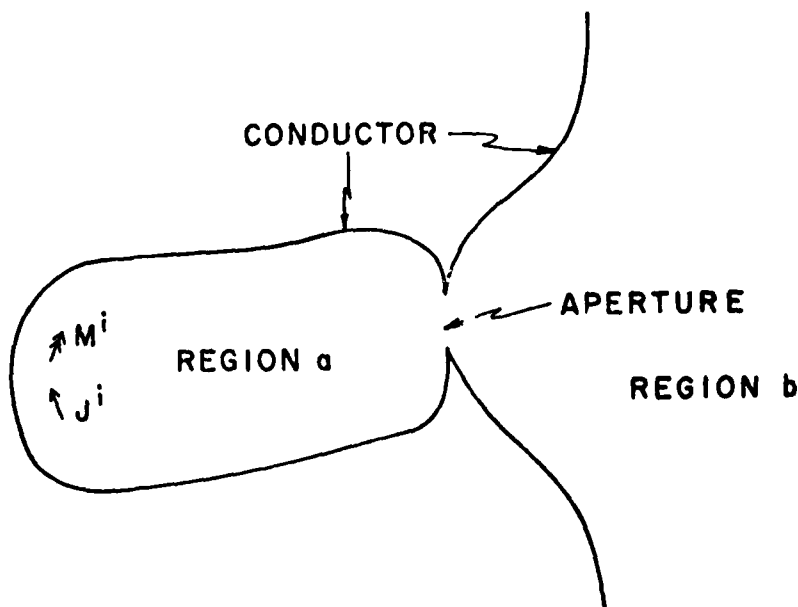
APERTURE

REGION b

Fig. 2-1. The general problem of two regions
coupled by an aperture.

problems as shown in Fig. 2-2. In region a, the field is produced by

the sources $\underline{J}^i$, $\underline{M}^i$, plus the equivalent magnetic current

$$\underline{M} = \underline{n} \times \underline{E} \tag{2-1}$$

over the aperture region, with the aperture covered by an electric con-

ductor. In region b, the field is produced by the equivalent magnetic

current $-\underline{M}$ over the aperture region, with the aperture covered by an

electric conductor. The fact that the equivalent current in region b is

the negative of that in region a ensures that the tangential component of

electric field is continuous across the aperture. The remaining boundary

condition to be applied is continuity of the tangential component of mag-

netic field across the aperture.

The tangential component of magnetic field in region a over the

aperture, denoted $\underline{H}_t^a$, is the sum of that due to the impressed sources,

denoted $\underline{H}_t^i$, plus that due to the equivalent source $\underline{M}$, denoted $\underline{H}_t^a(\underline{M})$,

that is

$$\underline{H}_t^a = \underline{H}_t^i + \underline{H}_t^a(\underline{M}) \tag{2-2}$$

Note that $\underline{H}_t^i$ and $\underline{H}_t^a(\underline{M})$ are both computed with a conductor covering the

aperture. A similar equation holds for region b, except that the equi-

valent source $-\underline{M}$ is the only source. Hence, the tangential component

of magnetic field in region b over the aperture is

$$\underline{H}_t^b = \underline{H}_t^b(-\underline{M}) = - \underline{H}_t^b(\underline{M}) \tag{2-3}$$

where $\underline{H}_t^b(\underline{M})$ is computed with a conductor covering the aperture. The last

equality in (2-3) is a consequence of the linearity of the $\underline{H}_t^b$ operator.

The true solution is obtained when $\underline{H}_t^a$ of (2-2) equals $\underline{H}_t^b$ of (2-3), or

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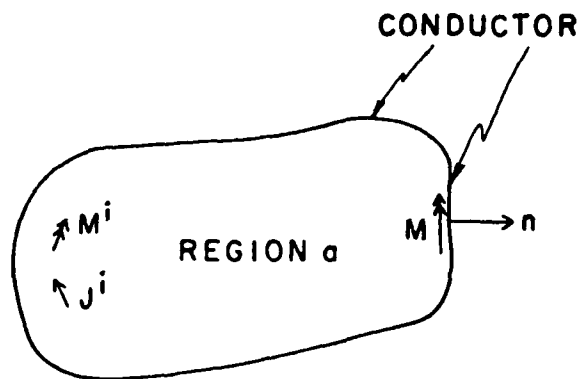
(a) EQUIVALENCE FOR REGION a.
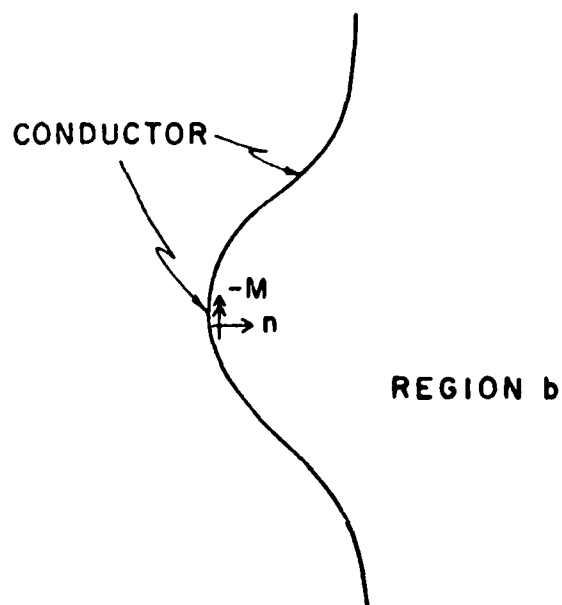
(b) EQUIVALENCE FOR REGION b.

Fig. 2-2. The original problem divided into two
equivalent problems.

$$-\underline{H}_t^a(\underline{M}) - \underline{H}_t^b(\underline{M}) = \underline{H}_t^i \tag{2-4}$$

This is the basic operator equation for determining the equivalent

magnetic current $\underline{M}$.

If (2-4) were satisfied exactly, we would have the true solution.

We use the method of moments   to   obtain an approximate solution.

Define a set of expansion functions $\{\underline{M}_n$, $n=1,2,\ldots,N\}$, and let

$$\underline{M} = \sum_n V_n \underline{M}_n \tag{2-5}$$

where the coefficients V_n are to be determined. Substitute (2-5) into

(2-4) and use the linearity of the $\underline{H}_t$ operators to obtain

$$-\sum_n V_n \underline{H}_t^a(\underline{M}_n) - \sum_n V_n \underline{H}_t^b(\underline{M}_n) = \underline{H}_t^i \tag{2-6}$$

Next, define a symmetric product

$$<A,B> = \iint_{\text{apert.}} \underline{A} \cdot \underline{B} \, ds \tag{2-7}$$

and a set of testing functions $\{\underline{W}_m$, $m=1,2,\ldots,N\}$, which may or may not be

equal to the expansion functions. We take the symmetric product of (2-6)

with each testing function $\underline{W}_m$, and use the linearity of the symmetric

product to obtain the set of equations

$$-\sum_n V_n <W_m, \; H_t^a(\underline{M}_n)> - \sum_n V_n <W_m, \; H_t^b(\underline{M}_n)> \; = \; <W_m, \; H_t^i> \tag{2-8}$$

$m=1,2,\ldots,N$. Solution of this set of linear equations determines the

coefficients V_n and the magnetic current $\underline{M}$ according to (2-5). Once $\underline{M}$

is known, the fields and field-related parameters may be computed by

standard methods.

The above solution can be put into matrix notation as follows:

Define an aperture admittance matrix for region a as

$$[Y^a] = [<-W_m, H_t^a(\underline{M}_n)>]_{N \times N} \qquad (2-9)$$

and an aperture admittance matrix for region b as

$$[Y^b] = [<-W_m, H_t^b(\underline{M}_n)>]_{N \times N} \qquad (2-10)$$

The minus signs are placed in (2-9) and (2-10) on the basis of power considerations. Define a source vector

$$\vec{I}^i = [<W_m, H_t^i>]_{N \times 1} \qquad (2-11)$$

and a coefficient vector

$$\vec{V} = [V_n]_{N \times 1} \qquad (2-12)$$

Now the matrix equation equivalent to equations (2-8) is

$$[Y^a + Y^b]\vec{V} = \vec{I}^i \qquad (2-13)$$

This can be interpreted in terms of generalized networks as two networks
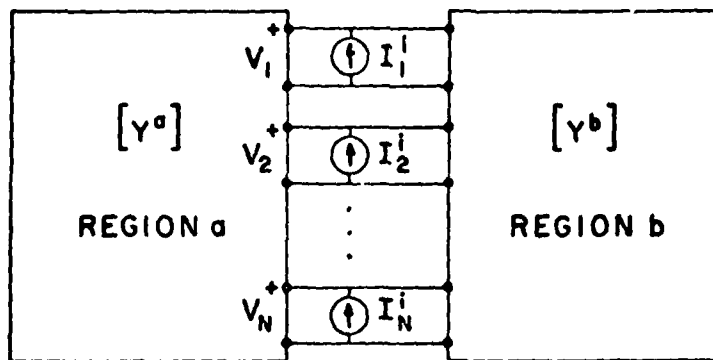
Fig. 2-3. The generalized network interpretation
of equation (13).

$[Y^a]$ and $[Y^b]$ in parallel with the current source $\vec{I}^i$, as shown in Fig. 2-3. The resultant voltage vector

$$\vec{V} = [Y^a + Y^b]^{-1}\vec{I}^i \qquad (2-14)$$

is then the vector of coefficients which gives $\underline{M}$ according to (2-5).

It is important to note that computation of $[Y^a]$ involves only region a, and computation of $[Y^b]$ involves only region b. Hence, we have divided the problem into two parts, each of which may be formulated independently. Once $[Y^a]$ is computed for one region, it may be combined with $[Y^a]$ for any other region, making it useful for a wide range of problems. For example, the same aperture admittance matrix for radiation into half-space would be useful for plane wave excitation of the aperture, waveguide excitation, and cavity excitation.

2-3 LINEAR MEASUREMENT

A linear measurement is defined as a number which depends linearly on the source. Examples of linear measurements are components of the field at a point, voltage along a given contour, and current crossing a given surface. Measurements made in region b will depend linearly only on the equivalent current $-\underline{M}$. Measurements made in region a will depend linearly on the impressed sources $\underline{J}^i$, $\underline{M}^i$, as well as on the equivalent current $\underline{M}$. We now illustrate these concepts with a particular example.

Consider the measurement (computation) of a component $\mathcal{H}_m$ of magnetic field at a point $\underline{r}_m$ in region b. It is known that this component can be obtained by placing a magnetic dipole $K\underline{\ell}_m$ at $\underline{r}_m$, and applying the reciprocity theorem to its field and to the original field, Section 3-8 of [1]. The original field in region b is given by the solution to Fig. 2-2b. The problem involving the magnetic dipole, called the adjoint problem, is shown in Fig. 2-4. Application of the reciprocity theorem
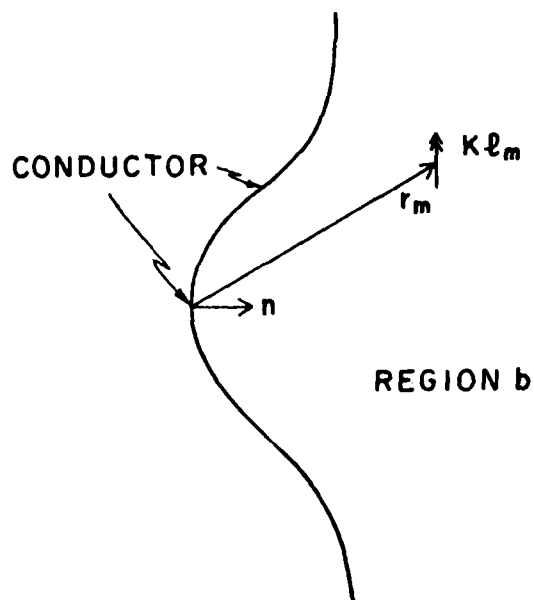
Fig. 2-4. The adjoint problem for
determining H_m at $\underline{r}_m$.

to these two cases yields

$$H_m K\ell_m = - \iint_{\text{apert.}} \underline{M} \cdot \underline{H}^m ds \qquad (2\text{-}15)$$

Here $\underline{H}^m$ is the magnetic field from $K\underline{\ell}_m$ in the presence of a complete
conductor, and H_m is the component in the direction of $K\underline{\ell}_m$ of the mag-
netic field at $\underline{r}_m$ due to $-\underline{M}$ in the presence of a complete conductor.
To evaluate (2-15), substitute for $\underline{M}$ from (2-5) and obtain

$$H_m K\ell_m = \sum_n V_n <-M_n, \ H^m> \qquad (2\text{-}16)$$

This can be written in matrix form as

$$H_m K\ell_m = \tilde{I}^m \ \vec{V} \qquad (2\text{-}17)$$

where $\tilde{I}^m$ is the transpose of a measurement vector

$$\vec{I}^m = [<-M_n, \; H^m>]_{N \times 1} \qquad (2\text{-}18)$$

Note that the elements of $\vec{I}^m$ are similar in form to those of $\vec{I}^i$ given by (2-11), except that $-\underline{M}_n$ replaces $\underline{W}_m$. The minus sign difference reflects the fact that the equivalent source in region b is $-\underline{M}$, in contrast to that in region a which is $+\underline{M}$. Now substitute (2-14) into (2-17) to obtain

$$H_m K\ell_m = \tilde{I}^m [Y^a + Y^b]^{-1} \vec{I}^i \qquad (2\text{-}19)$$

If the magnetic dipole is of unit moment, then (2-19) gives H_m at $\underline{r}_m$ directly.

Every linear measurement in region b will be of the form (2-19). For example, if a component of $\underline{E}$ at $\underline{r}_m$ were desired, we would place an electric dipole at $\underline{r}_m$ and *apply reciprocity*. In general, a linear measurement involves applying reciprocity to the original problem and to an adjoint problem. A determination of the sources of the adjoint problem is a part of the formulation of the problem.

If a linear measurement is made in region a, it will involve a contribution from the impressed sources $\underline{J}^i$, $\underline{M}^i$ added to that from the equivalent source $\underline{M}$. For example, instead of (2-19) we would have

$$H_m K\ell_m = H_m^i K\ell_m + \tilde{I}^m [Y^a + Y^b]^{-1} \vec{I} \qquad (2\text{-}20)$$

where H_m^i is the magnetic field from $\underline{J}^i$, $\underline{M}^i$ in the presence of a complete conductor. Also, in region a we would define the measurement vector to be

$$\vec{I}^m = [<M_n, \; H^m>]_{N \times 1} \qquad (2\text{-}21)$$

instead of (2-18), because the equivalent sources are $+\underline{M}$ in region a in

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contrast to $-\underline{M}$ in region b. Note that it is the difference field $\underline{H}-\underline{H}^1$ in region a (due to $\underline{M}$) that is directly analogous to the transmitted field $\underline{H}$ in region b (due to $-\underline{M}$).

2-4   TRANSMITTED POWER

A quadratic measurement is one which depends quadratically on the sources. Examples of quadratic measurements are components of the Poynting vector at a point, power crossing a given surface, and energy within a given region. A particular quadratic measurement of considerable interest is the power transmitted through the aperture, which we now consider.

The complex power P_t transmitted through the aperture is basically

$$P_t = \iint\limits_{\text{apert.}} \underline{E} \times \underline{H}^* \cdot \underline{n} \, ds \qquad (2-22)$$

where the asterisk denotes complex conjugate. Substituting from (2-1), we have

$$P_t = \iint\limits_{\text{apert.}} \underline{M} \cdot \underline{H}^* ds \qquad (2-23)$$

This involves only the tangential component of $\underline{H}$, which in region b we denoted by $H_t^b(-\underline{M})$. For $\underline{M}$ we use the linear combination (2-5) and obtain

$$\underline{H}_t^b(-\underline{M}) = - \sum_n V_n \underline{H}_t^b(\underline{M}_n) \qquad (2-24)$$

Substituting this for $\underline{H}$ and (2-5) for $\underline{M}$ into (2-23), we obtain

$$P_t = - \sum_m \sum_n V_m V_n^* \iint\limits_{\text{apert.}} \underline{M}_m \cdot \underline{H}_t^{b*}(\underline{M}_n) ds \qquad (2-25)$$

If $\underline{M}_m$ are real, the conjugate operations can be taken outside the

integrals. Moreover, if $\underline{M}_m = \underline{W}_m$ (Galerkin's method), then the negative of the integrals in (2-25) are Y_{mn}^{b*} as defined by (2-10), and

$$P_t = \sum_m \sum_n V_m V_n^* \, Y_{mn}^{b*} \tag{2-26}$$

This can be written in matrix form as

$$P_t = \tilde{V} [Y^b]^* \vec{V}^* \tag{2-27}$$

Note that this is the usual formula for power into network $[Y^b]$ of Fig. 2-3.

2-5 DISCUSSION

The basic formulation for two-region aperture problems is given in this chapter. Only a single aperture is considered explicitly, but the extension to multiple apertures is straightforward. In Chapter III, we apply the formulation to the problem of apertures in plane conducting screens. In Chapter IV, we apply it to waveguide-fed apertures. In Chapter V, the formulation is specialized to cavity-backed apertures. Chapter VI considers the case of a narrow slot in a thick conducting plane. Chapter VII considers the case of an aperture in a plane screen backed by a conducting body. Chapter VIII considers the equivalent circuit for coupling through an aperture to a long wire. There are, of course, infinitely many geometries that can be considered, and this report treats only a representative cross section of these possibilities. The basic principles remain the same regardless of the type of problem being considered.

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     Hill Book Company, New York, 1961.

Chapter III

APERTURES IN PLANE CONDUCTORS

3-1 GENERAL THEORY

Consider a conducting plane covering the z=0 plane except for an aperture, as shown in Fig. 3-1. The two regions $z > 0$ and $z < 0$ are identical-half spaces, and hence their aperture admittance matrices are the same. Therefore, we let

$$[Y^a + Y^b] = 2[Y^{hs}] \qquad (3-1)$$

where $[Y^{hs}]$ denotes the aperture admittance for the aperture opening into half space, say $z > 0$. When the aperture is covered by a conductor,
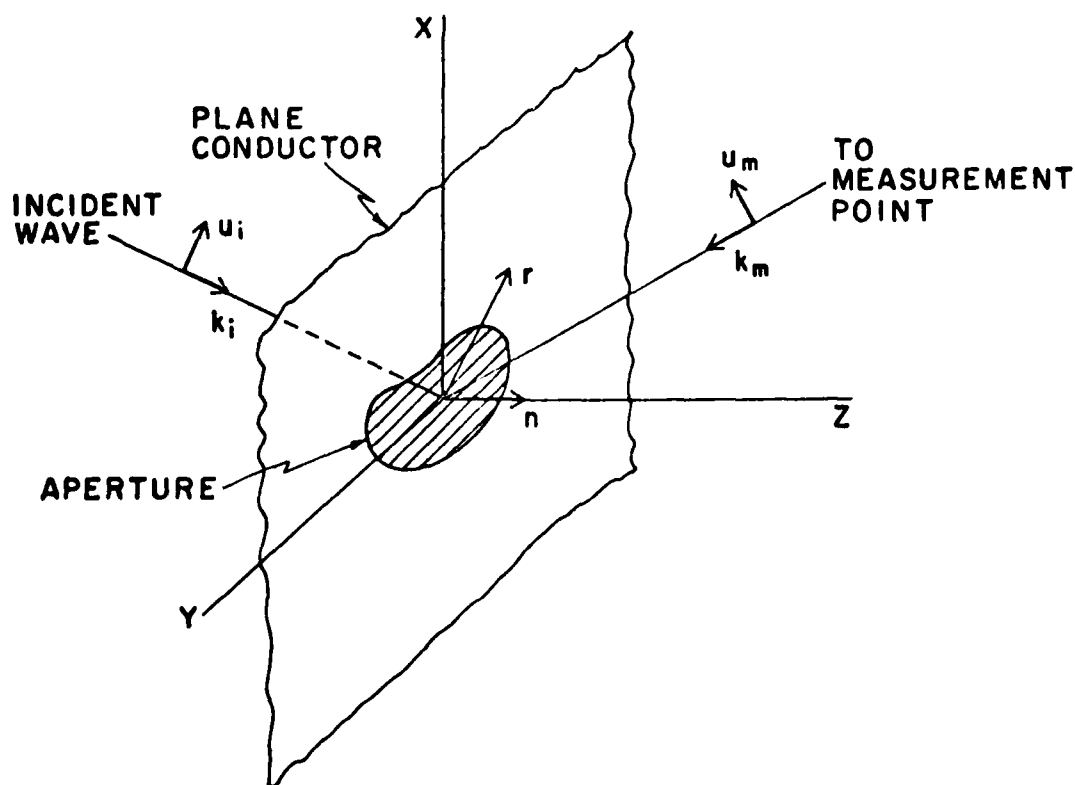
Fig. 3-1. Aperture in a plane conductor.

the $z = 0$ plane is a complete conducting plane, and image theory applies. The magnetic current expansion functions are on the surface of the $z = 0$ plane. Their images are equal to them and are also on the $z = 0$ plane, according to section 3-6 of [1]. The result is that $[Y^{hs}]$ is the admittance matrix obtained using expansion functions $2\underline{M}_n$ radiating into free space everywhere. This problem is dual to that for the impedance matrix of a plane conductor, a problem considered recently in the literature [2].

The original excitation of the aperture is by the impressed sources $\underline{J}^i$, $\underline{M}^i$ in the region $z < 0$. The impressed field $\underline{H}_t^i$ used in the operator equation (2-4) is the tangential magnetic field due to $\underline{J}^i$, $\underline{M}^i$ with the aperture covered by a conductor (Fig. 2-2a). In this case the $z = 0$ plane is a complete conductor, and image theory again applies. The result is that the tangential component of $\underline{H}$ over the $z = 0$ plane when it is covered by a conductor is just twice what it is for the same sources in free space. Hence,

$$\underline{H}_t^i = 2\underline{H}_t^{io} \tag{3-2}$$

where $\underline{H}_t^{io}$ is the tangential component of the magnetic field over the aperture due to the sources $\underline{J}^i$, $\underline{M}^i$ in free space. The components of the excitation vector $\vec{I}^i$ defined by (2-11) are now

$$I_m^i = 2 \iint\limits_{\text{apert.}} \underline{W}_m \cdot \underline{H}_t^{io} ds \tag{3-3}$$

where $\underline{W}_m$ is the mth testing function.

A case of special interest is that of plane wave excitation. A unit plane wave is given by

$$\underline{H}^{io} = \underline{u}_1 e^{-j\underline{k}_1 \cdot \underline{r}} \tag{3-4}$$

where $\underline{u}_i$ is a unit vector specifying the direction of $\underline{H}^{io}$, $\underline{k}_i$ is the propagation vector of magnitude $2\pi/\lambda$ and pointing in the direction of propagation, and $\underline{r}$ is the radius vector to an arbitrary field point. These vectors are shown in Fig. 3-1. The components (3-3) of the plane-wave excitation vector are then

$$P_m^i = 2 \iint\limits_{apert.} \underline{W}_m \cdot \underline{u}_i \, e^{-j\underline{k}_i \cdot \underline{r}} \, ds \qquad (3\text{-}5)$$

The symbol $\overrightarrow{P}^i$ has been used for this particular vector to distinguish it from the more general excitation vector (3-3).

Similar simplifications apply to the adjoint (measurement) problem. For the evaluation of a component of magnetic field at a point $\underline{r}_m$, a magnetic dipole $K\ell_m$ is placed at the measurement point $\underline{r}_m$. This radiates in the presence of a complete conductor over the $z = 0$ plane, and hence, analogous to (3-2), we have

$$\underline{H}_t^m = 2\underline{H}_t^{mo} \qquad (3\text{-}6)$$

Here $\underline{H}_t^m$ denotes the tangential component of $\underline{H}$ over the aperture from $K\ell_m$ when the $z = 0$ plane is covered by a conductor, and $\underline{H}_t^{mo}$ denotes that from $K\ell_m$ when it radiates into free space. The components of the measurement vector $\overrightarrow{I}^m$ defined by (2-18) are now

$$I_n^m = -2 \iint\limits_{apert.} \underline{M}_n \cdot \underline{H}_t^{mo} ds \qquad (3\text{-}7)$$

where $\underline{M}_n$ is the nth expansion function.

A case of special interest is that of far-field measurement. This is obtained by a procedure dual to that used for radiation and scattering from conducting wires [3]. To obtain a component of H

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on the radiation sphere, we take a source $K\underline{\ell}_m$ perpendicular to $\underline{r}_m$
and let $r_m \to \infty$. At the same time we adjust $K\ell_m$ so that it produces
a unit plane wave in the vicinity of the origin. The required
dipole moment is given by

$$\frac{1}{K\ell_m} = \frac{-j\omega\varepsilon}{4\pi r_m} e^{-jkr_m} \tag{3-8}$$

and the plane wave field it produces in the vicinity of the origin is

$$\underline{H}^{mo} = \underline{u}_m e^{-j\underline{k}_m \cdot \underline{r}} \tag{3-9}$$

Here $\underline{u}_m$ is a unit vector in the direction of $\underline{H}^{mo}$, $\underline{k}_m$ is the propaga-
tion vector, and $\underline{r}$ is the radius vector to an arbitrary field point.
Again these vectors are shown in Fig. 3-1. The components (3-7) of the
far-field measurement vector are then

$$P^m_n = -2 \iint\limits_{\text{apert.}} \underline{M}_n \cdot \underline{u}_m e^{-j\underline{k}_m \cdot \underline{r}} ds \tag{3-10}$$

The symbol $\vec{P}^m$ is used for this particular measurement vector to
distinguish it from the more general measurement vector (3-7). The
far-zone magnetic field is now given by (2-19) with $K\ell_m$ given by (3-8),
$\vec{I}^m = \vec{P}^m$, $\vec{I}^i = \vec{P}^i$, and $[Y^a + Y^b]$ given by (3-1). Hence

$$H_m = \frac{-j\omega\varepsilon}{8\pi r_m} e^{-jkr_m} \vec{P}^m [Y^{hs}]^{-1} \vec{P}^i \tag{3-11}$$

The usual two radiation components H_θ and H_ϕ are obtained by orienting
$K\underline{\ell}_m$ in the θ and ϕ directions, respectively.

3-2 TRANSMISSION PARAMETERS

A parameter sometimes used to express the transmission character-istics of an aperture is the transmission cross section τ. It is de-fined as that area for which the incident wave contains sufficient power to produce the radiation field H_m by omnidirectional radiation over half space. For unit incident magnetic field, this is

$$\tau = 2\pi r_m^2 |H_m|^2 \tag{3-12}$$

Substituting from (3-11), we obtain

$$\tau = \frac{\omega^2 \varepsilon^2}{32\pi} |\tilde{P}^m [Y^{hs}]^{-1} \vec{P}^i|^2 \tag{3-13}$$

Note that τ depends upon the polarization and direction of the incident wave (via $\vec{P}^i$), and upon the polarization measured and direction to the measurement point (via $\tilde{P}^m$).

Another parameter used to express the transmission characteristics of an aperture is the transmission coefficient T, defined as

$$T = \frac{P_{trans.}}{P_{inc.}} \tag{3-14}$$

where $P_{trans.}$ is the time-average power transmitted by the aperture, and $P_{inc.}$ is the free space power incident on the aperture. The inci-dent power is

$$P_{inc.} = \eta S \cos \theta_{inc.} \tag{3-15}$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of free space, S is the aperture area, and $\theta_{inc.}$ is the angle between $\underline{k}_i$ and $\underline{n}$. The trans-mitted power is

$$P_{trans} = \text{Re}(P_t) \tag{3-16}$$

where $\text{Re}(P_t)$ denotes the real part of P_t, given by (2-27) so that

$$T = \frac{1}{\eta S \cos \theta_{inc.}} \text{Re}(V[Y^{hs}]^* \vec{V}^*) \tag{3-17}$$

Note that T depends on both the direction of incidence and on the polarization of the incident wave.

Finally, because of symmetry about the $z = 0$ plane, the difference field $\underline{H}-\underline{H}^i$ which exists in the region $z < 0$ is simply related to the transmitted field which exists in the region $z > 0$. The difference field in the region $z < 0$ is produced by an equivalent current $\underline{M}$ on a plane conductor over the $z = 0$ plane. By image theory, it is also the field produced in the region $z < 0$ by the source $2\underline{M}$ in free space. Analogously, the transmitted field in the region $z > 0$ is produced by the source $-2\underline{M}$ in free space. Hence, the difference field in the region $z < 0$ and the negative of the transmitted field in the region $z > 0$ are both produced by the same magnetic current $2\underline{M}$ radiating in free space.

3-3 ADMITTANCE MATRIX

If an admittance matrix [Y] is defined by $[Y] = [Y^a + Y^b]$, then, according to (3-1) and (2-10), the ij-th element of [Y] is given by

$$Y_{ij} = (Y^a + Y^b)_{ij} = -4<W_i, H(\underline{M}_j)> \tag{3-18}$$

where $\underline{H}(\underline{M}_j)$ is the magnetic field produced by $\underline{M}_j$ radiating in free space. In view of Sections 1-3 and 1-4, the magnetic field $\underline{H}(\underline{M}_j)$ can be expressed in terms of an electric vector potential $\underline{F}_j$ and magnetic scalar potential ψ_j as [4]

$$\underline{H}(\underline{M}_j) = -j\omega\underline{F}_j - \underline{\nabla}\psi_j \tag{3-19}$$

where

$$\underline{F}_j = \frac{\varepsilon}{4\pi} \iint\limits_{apert.} \underline{M}_j \frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} ds' \tag{3-20}$$

$$\psi_j = \frac{1}{4\pi\mu} \iint\limits_{apert.} m_j \frac{e^{-jk|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} ds' \tag{3-21}$$

$$m_j = \frac{\underline{\nabla} \cdot \underline{M}_j}{-j\omega} \tag{3-22}$$

where $\underline{r}$ and $\underline{r}'$ are respectively the vectors to the field and source points in the aperture. Substituting (2-7) and (3-19) into (3-18), we obtain

$$Y_{ij} = 4 \iint\limits_{apert.} \underline{W}_i \cdot (j\omega\underline{F}_j + \underline{\nabla}\psi_j)ds . \tag{3-23}$$

If the component of $\underline{W}_i$ normal to the rim of the aperture vanishes on the rim of the aperture, then

$$\iint\limits_{apert.} \underline{\nabla} \cdot (\phi_j \underline{W}_i) ds = 0$$

The above equation can be rewritten as

$$\iint\limits_{apert.} \underline{W}_i \cdot \underline{\nabla}\phi_j ds + \iint\limits_{apert.} \phi_j \underline{\nabla} \cdot \underline{W}_i ds = 0 \tag{3-24}$$

Hence, (3-23) becomes

$$Y_{ij} = 4j\omega \iint\limits_{apert.} (\underline{F}_j \cdot \underline{W}_i + \psi_j m_i) ds \tag{3-25}$$

where

$$m_i = \frac{\underline{\nabla} \cdot \underline{W}_i}{-j\omega} \tag{3-26}$$

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We must now consider a specific problem in order to choose appropriate
expansion and testing functions.

3-4 THE RECTANGULAR APERTURE

The geometry and coordinate system for the rectangular aperture
in a conducting plane is shown in Fig. 3-2. For this problem, we
choose the set of testing functions $\underline{W}_i$ equal to the set of expansion
functions $\underline{M}_j$. The rectangular aperture $0 \leq x \leq L_x \Delta x$, $0 \leq y \leq L_y \Delta y$
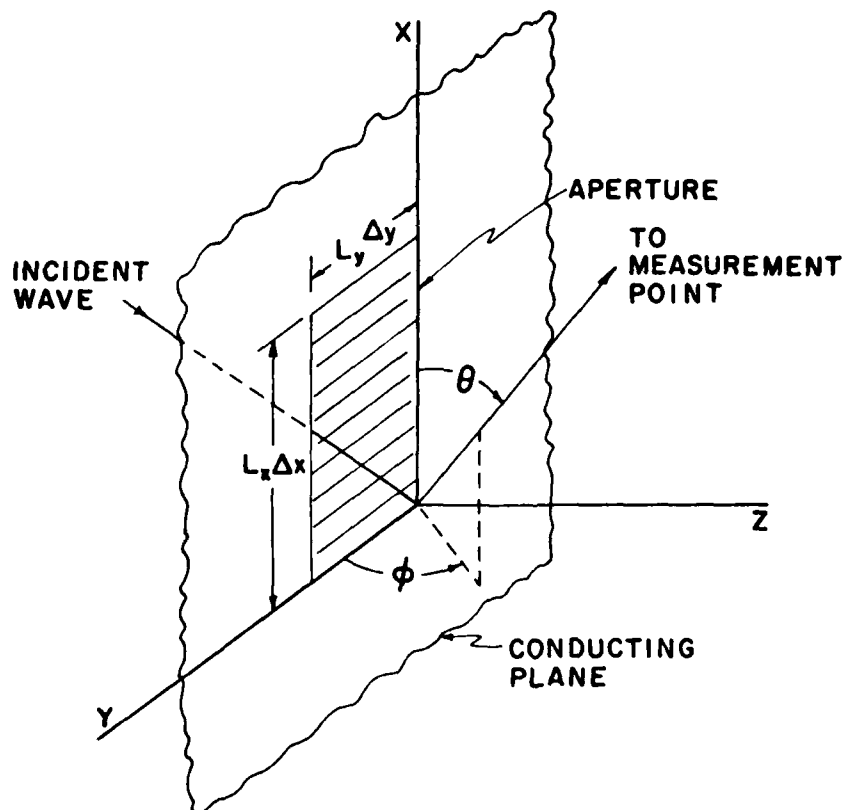

Fig. 3-2. Rectangular aperture in a conducting
plane.

where L_x and L_y are integers is divided into rectangular subareas of length Δx in x and Δy in y. The set $\underline{M}_j$ of expansion functions is split into a set $\underline{M}_j^x$ of x directed magnetic currents and a set $\underline{M}_j^y$ of y directed magnetic currents defined by

$$\underline{M}_{p+(q-1)(L_x-1)}^x = \hat{\underline{x}} \, T_p^x(x) \, P_q^y(y), \quad \begin{cases} p = 1,2,\ldots L_x-1 \\ \\ q = 1,2,\ldots L_y \end{cases} \quad (3\text{-}27)$$

$$\underline{M}_{p+(q-1)L_x}^y = \hat{\underline{y}} \, T_q^y(y) \, P_p^x(x) \, , \quad \begin{cases} p = 1,2,\ldots L_x \\ \\ q = 1,2,\ldots L_y-1 \end{cases} \quad (3\text{-}28)$$

where $\hat{\underline{x}}$ and $\hat{\underline{y}}$ are unit vectors. $T_p^x(x)$ and $T_q^y(y)$ are triangle functions defined by

$$T_p^x(x) = \begin{cases} \dfrac{x - (p-1)\Delta x}{\Delta x} & (p-1)\,\Delta x \leq x \leq p\Delta x \\ \\ \dfrac{(p+1)\Delta x - x}{\Delta x} & p\Delta x \leq x \leq (p+1)\Delta x \\ \\ 0 & |x - p\Delta x| \geq \Delta x \end{cases} \quad (3\text{-}29)$$

$$T_q^y(y) = \begin{cases} \dfrac{y - (q-1)\Delta y}{\Delta y} & (q-1)\Delta y \leq y \leq q\Delta y \\ \\ \dfrac{(q+1)\Delta y - y}{\Delta y} & q\Delta y \leq y \leq (q+1)\Delta y \\ \\ 0 & |y - q\Delta y| \geq \Delta y \end{cases} \quad (3\text{-}30)$$

and $P_p^x(x)$ and $P_q^y(y)$ are pulse functions defined by

$$P_p^x(x) = \begin{cases} 1 & (p-1)\Delta x \leq x < p\Delta x \\ \\ 0 & \text{all other } x \end{cases} \quad (3\text{-}31)$$

$$P_q^y(y) = \begin{cases} 1 & (q-1)\Delta y \leq y < q\Delta y \\ \\ 0 & \text{all other } y \end{cases} \quad (3\text{-}32)$$

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The magnetic charge sheets, say m_j^x and m_j^y associated with $\underline{M}_j^x$ and $\underline{M}_j^y$ are obtained from (3-22) as

$$m_{p+(q-1)(L_x-1)}^x = \frac{(P_p^x(x) - P_{p+1}^x(x))P_q^y(y)}{-j\omega\Delta x} \qquad (3-33)$$

$$m_{p+(q-1)L_x}^y = \frac{(P_q^y(y) - P_{q+1}^y(y))P_p^x(x)}{-j\omega\Delta y} \qquad (3-34)$$

Introduction of the two types of expansion functions $\underline{M}_j^x$ and $\underline{M}_j^y$ and the two types of testing functions $\underline{M}_i^x$ and $\underline{M}_i^y$ into (3-25) gives rise to four Y submatrices defined by

$$Y_{ij}^{uv} = 4j\omega \iint\limits_{apert.} (\underline{F}_j^v \cdot \underline{M}_i^u + \psi_j^v m_i^u)ds \begin{cases} u = x,y \\ v = x,y \end{cases} \qquad (3-35)$$

The mathematical details and approximations for numerically evaluating (3-35) can be found in a research report [5].

3-5 PLANE WAVE EXCITATION AND MEASUREMENT VECTORS

The plane wave excitation vector $\vec{P}^i$ of (3-5) and the plane wave measurement vector $\vec{P}^m$ of (3-10) are of the same form except for a minus sign. We therefore need to evaluate only one of them, say the measurement vector $\vec{P}^m$. We specialize it to four principal plane patterns as

$$(P_i^{mu})_{\theta y} = -2 \iint\limits_{apert.} \underline{M}_i^u \cdot \hat{\theta}\, e^{jkx \cos\theta}\, dxdy \qquad (3-36)$$

$$(P_i^{mu})_{yy} = -2 \iint\limits_{apert.} \underline{M}_i^u \cdot \hat{y}\, e^{jkx \cos\theta}\, dxdy \qquad (3-37)$$

$$(P_i^{mu})_{\phi x} = -2 \iint_{apert.} \underline{M}_i^u \cdot \hat{\underline{\phi}} \; e^{jky \cos \phi} \; dxdy \qquad (3\text{-}38)$$

$$(P_i^{mu})_{xx} = -2 \iint_{apert.} \underline{M}_i^u \cdot \hat{\underline{x}} \; e^{jky \cos \phi} \; dxdy \qquad (3\text{-}39)$$

where u is either x or y. The superscript u is necessary because $\underline{M}_i$ has been split up into $\underline{M}_i^x$ and $\underline{M}_i^y$ of (3-27) and (3-28). In (3-36) to (3-39), $\hat{\underline{\theta}}$, $\hat{\underline{y}}$, $\hat{\underline{\phi}}$, and $\hat{\underline{x}}$ are unit vectors in the θ, y, ϕ, and x directions respectively where, as shown in Fig. 3-2, θ is measured from the positive x axis in the y = 0 plane and ϕ is measured from the positive y axis in the x = 0 plane. For measurement vectors, $0° \le \theta \le 180°$, $0° \le \phi \le 180°$. $(P_i^{mu})_{\theta y}$ is for a $\hat{\underline{\theta}}$ polarized measurement in the y = 0 plane, $(P_i^{mu})_{yy}$ is for a $\hat{\underline{y}}$ polarized measurement in the y = 0 plane, $(P_i^{mu})_{\phi x}$ is for a $\hat{\underline{\phi}}$ polarized measurement in the x = 0 plane, and $(P_i^{mu})_{xx}$ is for an $\hat{\underline{x}}$ polarized measurement in the x = 0 plane. Because our set of testing functions $\underline{W}_i$ is the same as the set of expansion functions $\underline{M}_j$, the plane wave excitation vector $\vec{P}^i$ of (3-5) is obtained by putting $180° \le \theta < 360°$, $180° \le \phi \le 360°$ in the negative of one of the equations (3-36) to (3-39).

Again the mathematical evaluation of (3-36) to (3-39) can be found in the report [5]. A computer program, complete with operating instructions, for computing the transmission through a rectangular slot in a conducting plane is given in Part Two of [5].

3-6 REPRESENTATIVE COMPUTATIONS

A number of representative computations using the above matrix solution is given in the report [5]. We summarize some of these results here.

The first computations were made for a narrow slot, of width $\lambda/20$ in the y direction of variable length L in the x direction. The far-zone quantity plotted was the transmission cross section, given by (3-12), where H_m is the component of magnetic field being considered. We use the notation:

$$\tau_{\theta y} = 2\pi r_m^2 |H_\theta|^2 \qquad \text{in the } y = 0 \text{ plane}$$

$$\tau_{xx} = 2\pi r_m^2 |H_x|^2 \qquad \text{in the } x = 0 \text{ plane.}$$

(3-40)

We choose $\Delta y = \lambda/20$ so that $q = 1$ in (3-27) and (3-28). In this case, there are no y directed magnetic current expansion functions (3-28). Thus, the matrix solution (2-5) for the magnetic current $\underline{M}$ is an x directed vector. As a result, $H_y = 0$ in the $y = 0$ plane and $H_\phi = 0$ in the $x = 0$ plane. In other words, for the case being considered, the components of $\underline{H}$ orthogonal to those in (3-40) are zero.

Figure 3-3 shows plots of $\tau_{\theta y}$ and τ_{xx} for x-directed slots of width $\lambda/20$ and length (a) $L = \lambda/4$, (b) $L = \lambda/2$, (c) $L = 3\lambda/4$, and (d) $L = \lambda$. In all cases the excitation was due to a plane wave normally incident on the conducting plane with the magnetic field in the x direction. Note the large transmission cross section for $L \approx \lambda/2$, case (b), due to the slot being near resonance. The plots of τ are of the same form as scattering cross section from the complementary conducting strips, as known from Babinet's principle.

Figure 3-4 shows plots of the equivalent magnetic current in the aperture region for the same slots. Since $\underline{M} = \hat{z} \times \underline{E}$, they are also plots of the tangential component of $\underline{E}$ in the slots. Again note the large value of $\underline{M}$ for the case $L \approx \lambda/2$, which is near resonance. Note also that,
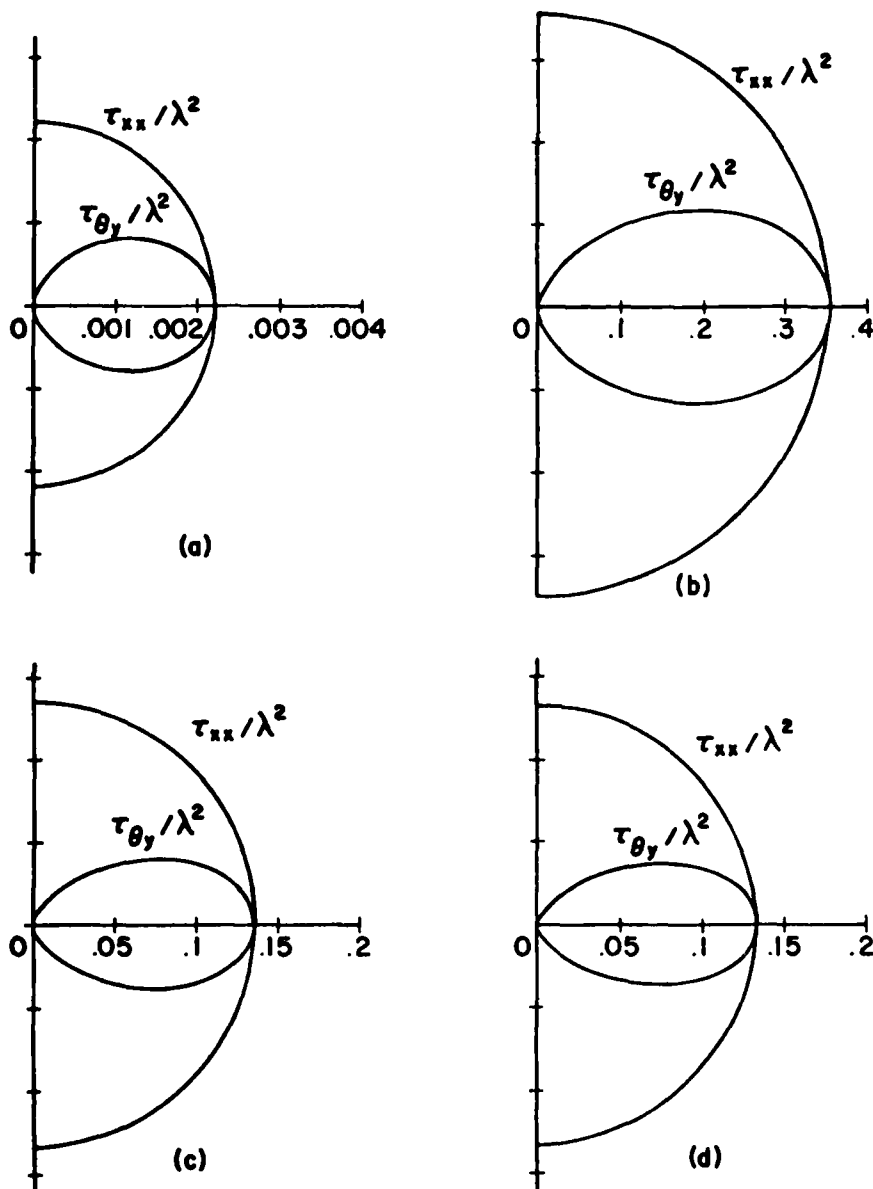
Fig. 3-3. Transmission cross section for slots of length L in the x
direction and width $\lambda/20$ in the y direction. (a) L = $\lambda/4$,
(b) L = $\lambda/2$, (c) L = $3\lambda/4$, (d) L = λ. Excitation is by a
plane wave normally incident on the conducting plane with
magnetic field in the x direction.
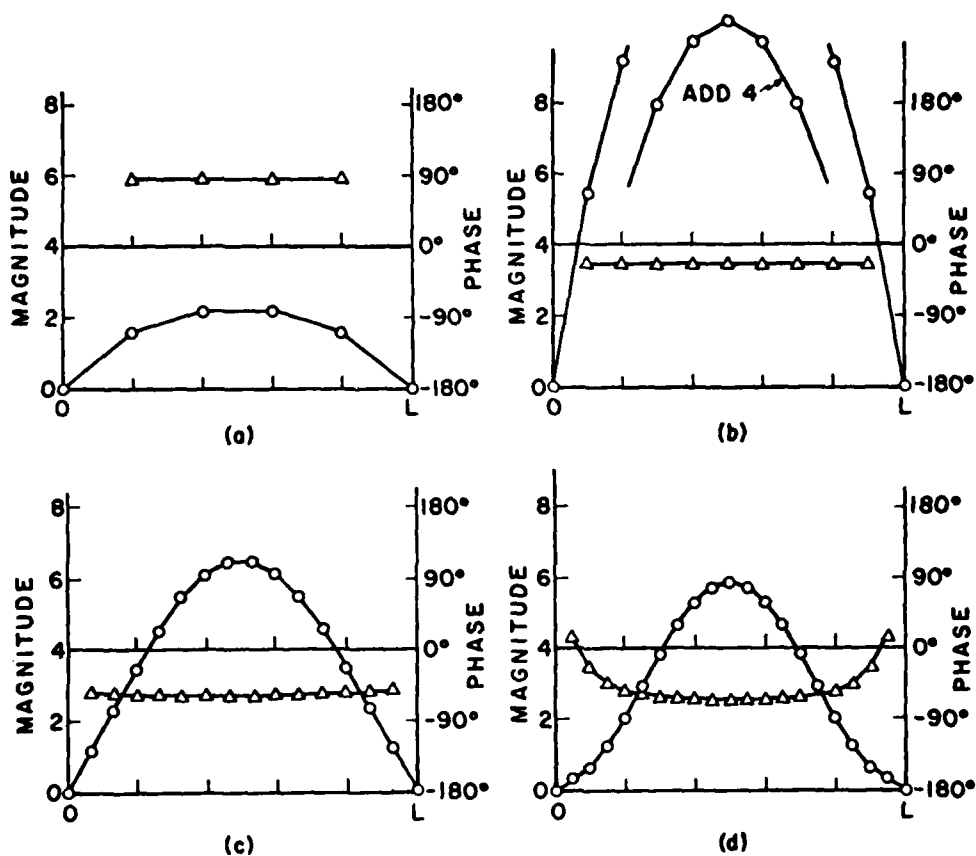
Fig. 3-4. Magnitude and phase of $|M/E^i|$, where M is the x-directed magnetic current and E^i is the incident electric field, for the same slots as for Fig. 3-3, (a) L = $\lambda/4$, (b) L = $\lambda/2$, (c) L = 3 /4, (d) L = λ. Circles denote magnitude, triangles denote phase.

for short slots ($L \leq 3\lambda/4$), the $\underline{M}$ is almost equiphasal and closely approximated by a half sine wave.

Next, computations were made to test the rate of convergence of the solution as the number of subsections was increased. A slot of width $\lambda/10$ and length 2λ was chosen for the study. Again the excitation is a plane wave normally incident on the conducting plane with the magnetic field in the x direction. Figure 3-5 shows plots of $\tau_{\theta y}$ and τ_{xx} for the cases (a) 39, (b) 19, (c) 9, and (d) 4 triangular expansion functions respectively. Note that the patterns (a) and (b) are essentially the same, and pattern (c) is only slightly different. They differ appreciably from (d), which results from only 4 expansion functions. The difference in the solutions as the number of expansion functions is decreased is better illustrated by plots of $\underline{M}$, as shown in Fig. 3-6. These are for the same cases as for Fig. 3-5. It can be seen clearly how the computed equivalent current in the slot region changes as the number of subsections is reduced. As a rule of thumb, for near-field quantities (such as $\underline{M}$) one should use subareas of length $\lambda/10$ or less and for far-field quantities (such as τ) length $\lambda/5$ or less.

3-7 DISCUSSION

The computer program, given in Part Two of [5], is written explicitly for rectangular apertures, but the formulas are valid for any aperture composed of rectangular subsections. Other apertures, such as L-shaped, T-shaped, square 0-shaped, etc., could be treated by appropriately changing the computer program. Apertures of arbitrary shape

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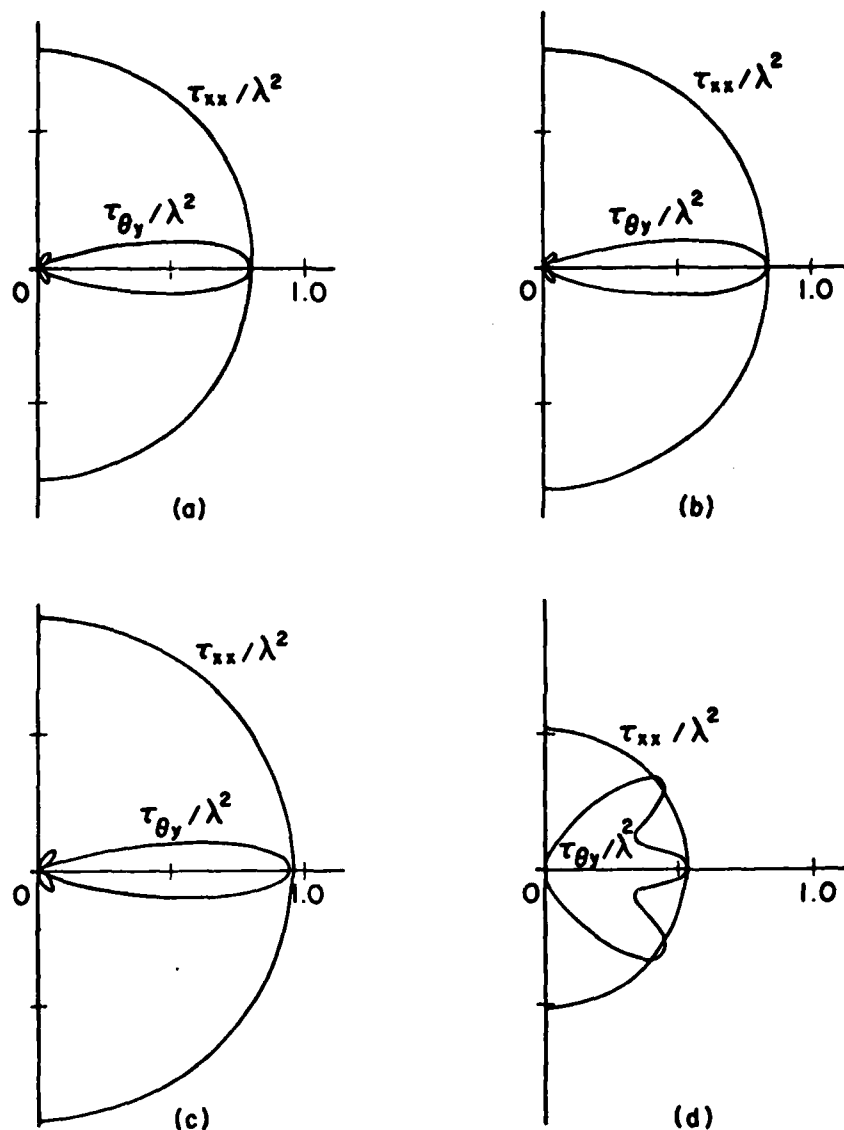
Fig. 3-5. Transmission cross section when the number of expansion func-
   tions is (a) 39, (b) 19, (c) 9, and (d) 4. Computations are
           for a slot of length 2λ in the x direction and width $\lambda/10$ in
           the y direction. Excitation is by a plane wave normally
           incident on the conducting plane with magnetic field in the
           x direction.

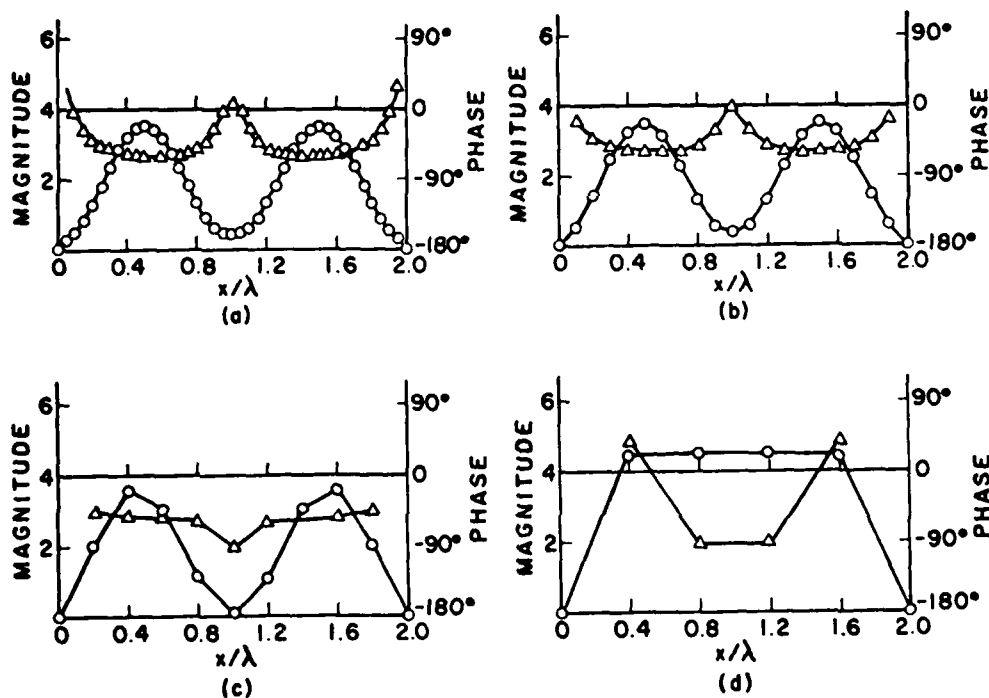Fig. 3-6. Magnitude and phase of $|M/E^i|$, where M is the x-directed magnetic current and E^i is the incident electric field, when the number of expansion functions is (a) 39, (b) 19. (c) 9, and (d) 4. Circles denote magnitude, triangles denote phase. Computations are for the same slot as for Fig. 3-5.

could be treated by approximating them by rectangular subsections. As with all moment solutions, the size of the apertures which can be treated depends upon the size of the matrix which can be computed and inverted. The examples indicate that the rectangular subsections should have side lengths not greater than 0.2 wavelengths for reasonable accuracy.

The aperture admittance matrix for radiation into half-space has application to any problem in which one region is bounded by a plane conductor, as shown in Chapter II. Hence, it can be used for a waveguide-fed aperture in a ground plane and for a cavity-backed aperture in a ground plane. These problems are considered in the next two chapters.

3-8   REFERENCES FOR CHAPTER III

[1]   R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill Book Co., New York, 1961.

[2]   N. N. Wang, J. H. Richmond, M. C. Gilreath, "Sinusoidal Reaction Formulation for Radiation and Scattering from Conducting Surfaces," IEEE Trans. on Antennas and Propagation, vol. AP-23, No. 3, pp. 376-382, May 1975.

[3]   R. F. Harrington, "Matrix Methods for Field Problems," Proc. IEEE, vol. 55, No. 2, pp. 136-149, February 1967.

[4]   C. H. Papas, Theory of Electromagnetic Wave Propagation, McGraw-Hill Book Co., New York, 1965, p. 23.

[5]   J. R. Mautz and R. F. Harrington, "Electromagnetic Transmission through a Rectangular Aperture in a Perfectly Conducting Plane," Report AFCRL-TR-76-0056, Air Force Cambridge Research Laboratories, Hanscom AFB, February 1976.

Chapter IV

WAVEGUIDE-FED APERTURES

4-1 GENERAL THEORY

Consider now a uniform waveguide feeding an aperture in a con-
ducting plane, as shown in Fig. 4-1. In general, the aperture may be
of different size and shape than the waveguide cross section. The half-
space region $z > 0$ is the same as in the previous problem, Fig. 3-1, and
the analysis of the preceding chapter applies. An analysis of the
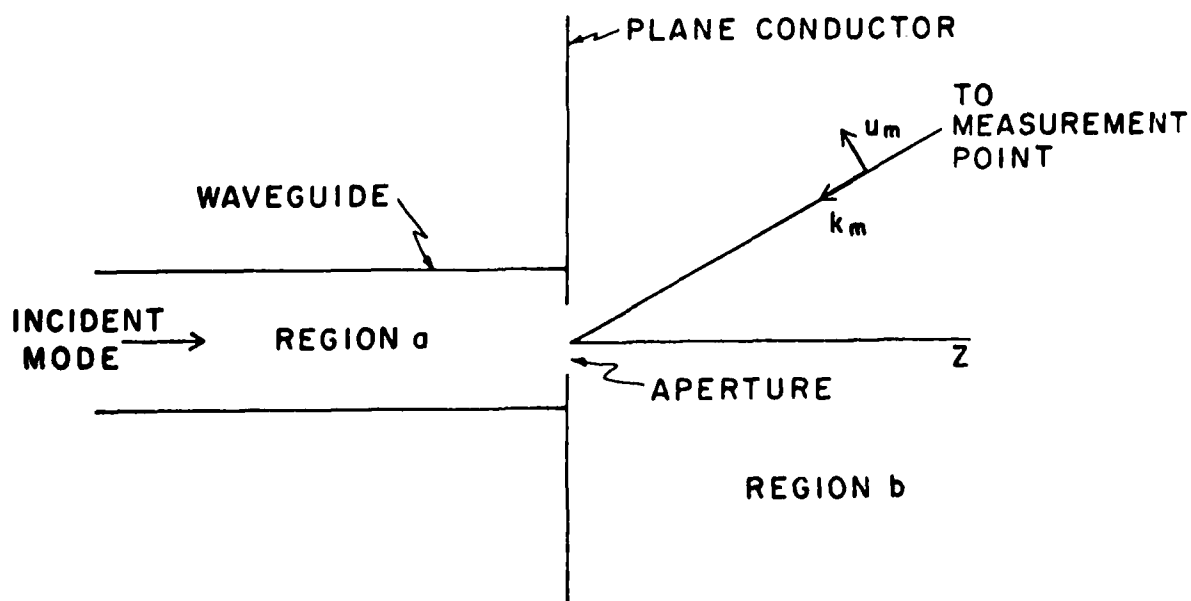waveguide region is given below.

Fig. 4-1. Waveguide-fed aperture in a conducting plane.

Let the excitation of the waveguide be a source which produces a single mode, of unit amplitude, incident on the aperture. This mode (usually the dominant mode) is denoted by the index o. The field transverse to the z-direction can then be expressed in modal form as (Sect. 8-1 of [1]).

$$\underline{E}_t = e^{-\gamma_o z}\, \underline{e}_o + \sum_i \Gamma_i e^{\gamma_i z}\, \underline{e}_i$$

$$\underline{H}_t = Y_o e^{-\gamma_o z}\, \underline{u}_z \times \underline{e}_o - \sum_i \Gamma_i Y_i\, e^{\gamma_i z}\, \underline{u}_z \times \underline{e}_i$$

$$(4-1)$$

It is assumed that all modes, TE and TM, are included in the summation. The γ_i are modal propagation constants

$$\gamma_i = \begin{cases} j\beta_i = jk\ \sqrt{1 - (\lambda/\lambda_i)^2} & \lambda < \lambda_i \\ \\ \alpha_i = k_i\ \sqrt{1 - (\lambda_i/\lambda)^2} & \lambda > \lambda_i \end{cases} \qquad (4-2)$$

where λ_i is the ith mode cut-off wavelength, and $k_i = 2\pi/\lambda_i$ is the ith mode cut-off wavenumber. The Y_i are the modal characteristic admittances

$$Y_i = \begin{cases} \gamma_i/j\omega\mu & , \qquad \text{TE modes} \\ \\ j\omega\varepsilon/\gamma_i & , \qquad \text{TM modes} \end{cases} \qquad (4-3)$$

Γ_o is the reflection coefficient for the oth mode, and Γ_i is the complex amplitude of the -z traveling component of the ith mode. The $\underline{e}_i$ are normalized modal vectors, so that the modal orthogonality relationships are

$$\iint\limits_{\text{guide}} \underline{e}_i \cdot \underline{e}_j\, ds = \begin{cases} 0 & i \neq j \\ \\ 1 & i = j \end{cases} \qquad (4-4)$$

where the integration is over the waveguide cross section.

To evaluate the aperture admittance (2-9) for the waveguide region, we consider a single expansion function $\underline{M}_n$ on the $z = 0$ plane in the waveguide region. The tangential field produced by $\underline{M}_n$ will be of the form (4-1), except that there is no incident wave. Hence, this field is

$$\underline{E}_t^a(\underline{M}_n) = \sum_i A_{ni} e^{\gamma_i z} \underline{e}_i$$

$$(4-5)$$

$$\underline{H}_t^a(\underline{M}_n) = -\sum_i A_{ni} Y_i e^{\gamma_i z} \underline{u}_z \times \underline{e}_i$$

where the A_{ni} are modal amplitudes. At $z = 0$ we have

$$\underline{M}_n = \underline{u}_z \times \underline{E}_t^a \Big|_{z=0} = \sum_i A_{ni} \underline{u}_z \times \underline{e}_i \qquad (4-6)$$

Multiply each side of this equation scalarly by $\underline{u}_z \times \underline{e}_j$ and integrate over the waveguide cross section, obtaining

$$\iint_{\text{guide}} \underline{M}_n \cdot \underline{u}_z \times \underline{e}_j ds = \sum_i A_{ni} \iint_{\text{guide}} (\underline{u}_z \times \underline{e}_i) \cdot (\underline{u}_z \times \underline{e}_j) ds \qquad (4-7)$$

By orthogonality (4-4), all terms of the summation are zero except the $i = j$ term. Hence,

$$A_{ni} = \iint_{\text{apert.}} \underline{M}_n \cdot \underline{u}_z \times \underline{e}_i ds \qquad (4-8)$$

We have replaced the integral over the waveguide cross section by one over the aperture, since $\underline{M}_n$ exists only in the aperture region. The elements of the aperture admittance matrix (2-9) are now given by

$$Y_{mn}^{wg} = -\iint_{\text{apert.}} \underline{W}_m \cdot \underline{H}_t^a(\underline{M}_n) ds \qquad (4-9)$$

where the superscript wg denotes waveguide. The $\underline{H}_t^a$ of (4-9) is given
by the second equation of (4-5) evaluated at $z = 0$, so that

$$Y_{mn}^{wg} = \sum_i A_{ni} Y_i \iint\limits_{\text{apert.}} \underline{W}_m \cdot \underline{u}_z \times \underline{e}_i ds \qquad (4-10)$$

Now define the constants

$$B_{mi} = \iint\limits_{\text{apert.}} \underline{W}_m \cdot \underline{u}_z \times \underline{e}_i ds \qquad (4-11)$$

which are similar in form to the A_{ni} of (4-8). The elements (4-10)
then are given by

$$Y_{mn}^{wg} = \sum_i A_{ni} B_{mi} Y_i \qquad (4-12)$$

Hence, all elements of the waveguide aperture admittance matrix $[Y^{wg}]$ are
linear combinations of the modal characteristic admittance Y_i. For
Galerkin's method, $\underline{W}_n = \underline{M}_n$ and $\underline{A}_{ni}$ and $\underline{B}_{ni}$ are equal.

We next evaluate the equivalent magnetic current $\underline{M}$, given by (2-5).
The incident field is given by the first term on the right-hand side of
(4-1). When the aperture is covered by a conductor, the waveguide is
terminated by a conducting plane. According to image theory, the tan-
gential magnetic field at $z = 0$ is then just twice the incident wave or

$$\underline{H}_t^i = 2Y_o \underline{u}_z \times \underline{e}_o \qquad (4-13)$$

This is the $\underline{H}_t^i$ used in (2-11) to evaluate the excitation vector $\vec{I}^i$.
Hence, the components of the excitation vector are

$$I_m^i = 2Y_o \iint\limits_{\text{apert.}} \underline{W}_m \cdot \underline{u}_z \times \underline{e}_o ds = 2Y_o B_{mo} \qquad (4-14)$$

The total aperture admittance matrix is

$$[Y^a + Y^b] = [Y^{wg} + Y^{hs}] \qquad (4\text{-}15)$$

where $[Y^{wg}]$ is the waveguide aperture admittance and $[Y^{hs}]$ is the half-space aperture admittance. The coefficient matrix $\vec{V}$ is given by (2-14) with the admittance matrix given by (4-15), or

$$\vec{V} = [Y^{wg} + Y^{hs}]^{-1}\vec{I}^i \qquad (4\text{-}16)$$

Finally, the equivalent magnetic current $\underline{M}$ is given by (2-5) where the coefficients V_n are the components of $\vec{V}$.

Once $\underline{M}$ is found, the modal amplitudes Γ_i in (4-1) can be evaluated from (2-1) and the orthogonality properties of the modes. From (2-1) and (4-1), we have

$$\underline{M} = \underline{u}_z \times \underline{E}_t \Big|_{z=0} = \underline{u}_z \times \underline{e}_o + \sum_i \Gamma_i \underline{u}_z \times \underline{e}_i \qquad (4\text{-}17)$$

Now multiply each side scalarly by $\underline{u}_z \times \underline{e}_j$ and integrate over the waveguide cross section. By the orthogonality relationships (4-4), all terms of the summation vanish except the term $i = j$. The result is

$$\iint_{\text{guide}} \underline{M} \cdot \underline{u}_z \times \underline{e}_i \, ds = \begin{cases} 1 + \Gamma_o & i = 0 \\ \\ \Gamma_i & i \neq 0 \end{cases} \qquad (4\text{-}18)$$

Here the integration over the guide can be changed to that over the aperture because $\underline{M} = 0$ except in the aperture. Substituting for $\underline{M}$ from (2-5) into (4-18), and using the definitions (4-8), we have

$$\sum_n V_n A_{no} = 1 + \Gamma_o$$

$$\sum_n V_n A_{ni} = \Gamma_i \quad i \neq 0 \qquad (4\text{-}19)$$

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Finally, by defining modal measurement vectors as

$$\vec{A}_i = [A_{ni}]_{N \times 1} \tag{4-20}$$

and using (4-16), we can write (4-19) as

$$1 + \Gamma_o = \tilde{A}_o [Y^{wg} + Y^{hs}]^{-1} \vec{I}^i \tag{4-21}$$

and, for $i \neq 0$,

$$\Gamma_i = \tilde{A}_i [Y^{wg} + Y^{hs}]^{-1} \vec{I}^i \tag{4-22}$$

The parameter of most interest is Γ_o, the reflection coefficient of the incident mode. This is often expressed in terms of an admittance

$$Y_{ap} = \frac{1 - \Gamma_o}{1 + \Gamma_o} Y_o \tag{4-23}$$

which is the equivalent aperture admittance seen by the incident mode.

The region $z > 0$ for the waveguide-fed aperture is the same half-space region as existed in the previous problem of an aperture in a conducting plane. Hence, evaluation of the fields in terms of $\underline{M}$ in this region is done in the same way as in Chapter III. For example, the $\underline{u}_m$ component of the far-zone magnetic field at a point $\underline{r}_m$ is given by

$$H_m = \frac{-j\omega\epsilon}{4\pi r_m} e^{-jkr_m} \tilde{P}^m [Y^{wg} + Y^{hs}]^{-1} \vec{I}^i \tag{4-24}$$

which is (3-11) with the term $[2Y^{hs}]^{-1}\vec{P}^i$ replaced by (4-16). The excitation vector $\vec{I}^i$ has elements given by (4-14), and the far-field measurement vector $\vec{P}^m$ has elements given by (3-10). The power gain pattern G is the ratio of the radiation intensity in a given direction to the radiation intensity which would exist if the total power $Re(P_t)$ were radiated

uniformly over half space, or

$$G = \frac{2\pi r_m^2 \eta |H_m|^2}{Re(P_t)} \qquad (4-25)$$

Substituting for H_m from (4-24), we have

$$G = \frac{\omega^2 \epsilon^2 \eta}{8\pi Re(P_t)} \; |\tilde{P}^m [Y^{wg} + Y^{hs}]^{-1} \; \vec{I}^i|^2 \qquad (4-26)$$

where P_t is given by (2-27). Note that this gain is a function of the $\underline{H}$ component measured, as well as direction to the field point.

4-2 APPLICATION TO A RECTANGULAR WAVEGUIDE

We now apply the general theory to a rectangular waveguide feeding a rectangular aperture in a conducting screen. Figure 4-2 shows the problem to be considered and defines the coordinates and parameters to be used. The perfectly conducting plate covers the entire $z = 0$ plane except for the aperture which is rectangular in shape with side lengths $L_x \Delta x$ and $L_y \Delta y$ in the x and y directions respectively. L_x and L_y are positive integers and $L_x \geq 2$. The aperture is fed by a rectangular waveguide. The excitation of the waveguide is a source which produces one mode, of unit amplitude, which travels toward the aperture.

The general method of solution discussed in Chapter II is to cover the aperture with a perfect electric conductor, to place magnetic current sheets $+\underline{M}$ and $-\underline{M}$ respectively on the left-hand and right-hand sides of this conductor, to obtain an integral equation for $\underline{M}$ by equating the tangential magnetic fields on both sides of this conductor, and to solve this integral equation using the method of moments. The testing functions are the same as the expansion functions for $\underline{M}$ and are denoted by $\underline{M}_i'$.
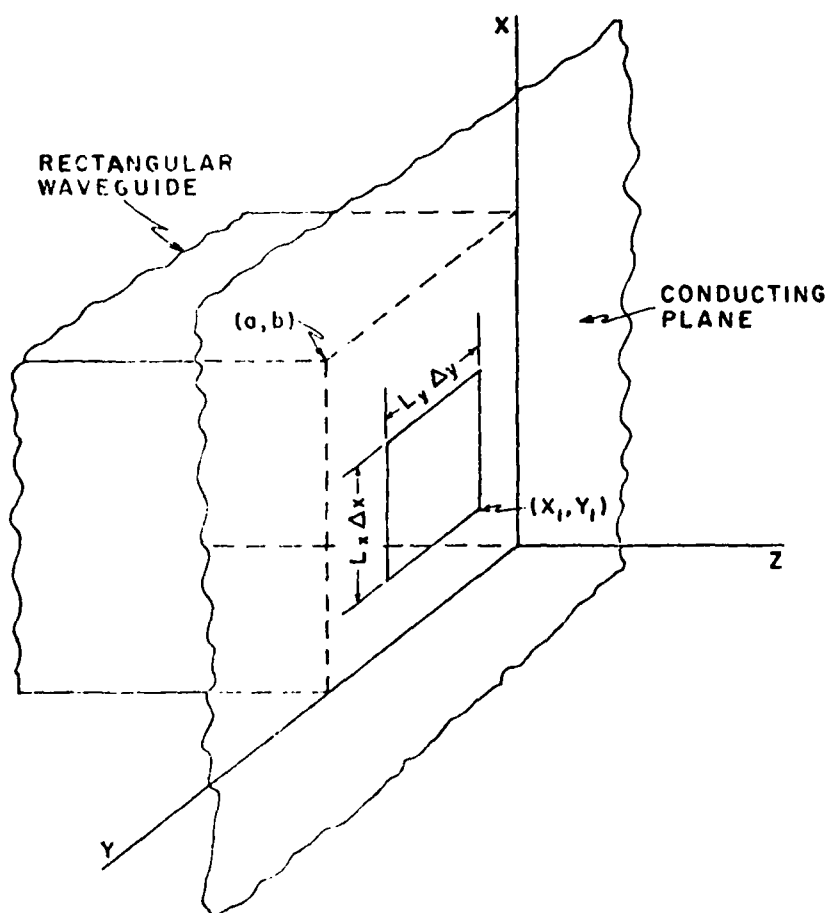
Fig. 4-2. A rectangular waveguide radiating through a rectangular
aperture into half-space bounded by an electric conductor.

Each $\underline{M}_i$ is a triangle in the direction of current flow and a pulse in

the direction perpendicular to current flow.

Expression (4-8) for A_{ij} requires a knowledge of the expansion

functions $\underline{M}_i$ and waveguide modes $\underline{e}_j$. The set $\underline{M}_i$ of expansion functions

is split into a set $\underline{M}_i^x$ of x directed magnetic currents and a set $\underline{M}_i^y$ of

y directed magnetic currents defined by

$$\underline{M}^x_{p+(q-1)(L_x-1)} = \hat{\underline{x}} \ T^x_p(x-x_1) \ P^y_q(y-y_1) \qquad \begin{cases} p=1,2,\ldots L_x-1 \\ \\ q=1,2,\ldots L_y \end{cases} \qquad (4\text{-}27)$$

$$\underline{M}^y_{p+(q-1)L_x} = \hat{\underline{y}} \ T^y_q(y-y_1) \ P^x_p(x-x_1) \qquad \begin{cases} p=1,2,\ldots L_x \\ \\ q=1,2,\ldots L_y-1 \end{cases} \qquad (4\text{-}28)$$

where $T^x_p(x)$ and $T^y_q(y)$ are triangle functions defined by (3-29) and (3-30), and $P^x_p(x)$ and $P^y_q(y)$ are the pulse functions defined by (3-31) and (3-32). An evaluation of the matrix $[Y^{wg}]$ is given in detail in [2]. An evaluation of the matrix $[Y^{hs}]$ is the same as that for the rectangular aperture, given in Sect. 3-4 of the preceding chapter. The listing and documentation of a computer program for the problem of Fig. 4-2 is given in Part II of the report [2].

4-3 SAMPLE COMPUTATIONS

In this section we give some representative computations for the aperture of Fig. 4-2. Figure 4-3 shows computed results for a rectangular waveguide of dimensions λ by $\lambda/2$ radiating into half space through a narrow centered rectangular slot of dimensions λ by $\lambda/10$, that is, $a = \lambda$ and $b = \lambda/2$. Figure 4-3(a) shows the x-component of equivalent magnetic current, which is also equal to the y-component of tangential $\underline{E}$ field in the slot. No y-component of magnetic current was obtained because only one pulse in y was used. $\underline{M}$ is normalized with respect to

$$\sqrt{\frac{1}{ab} \iint\limits_{\text{guide}} |\underline{e}_o|^2 \ dx \ dy} \qquad (4\text{-}33)$$

where the integral is over the waveguide cross section. In other words, the normalization factor is the root-mean-square value of the $\underline{E}$ field
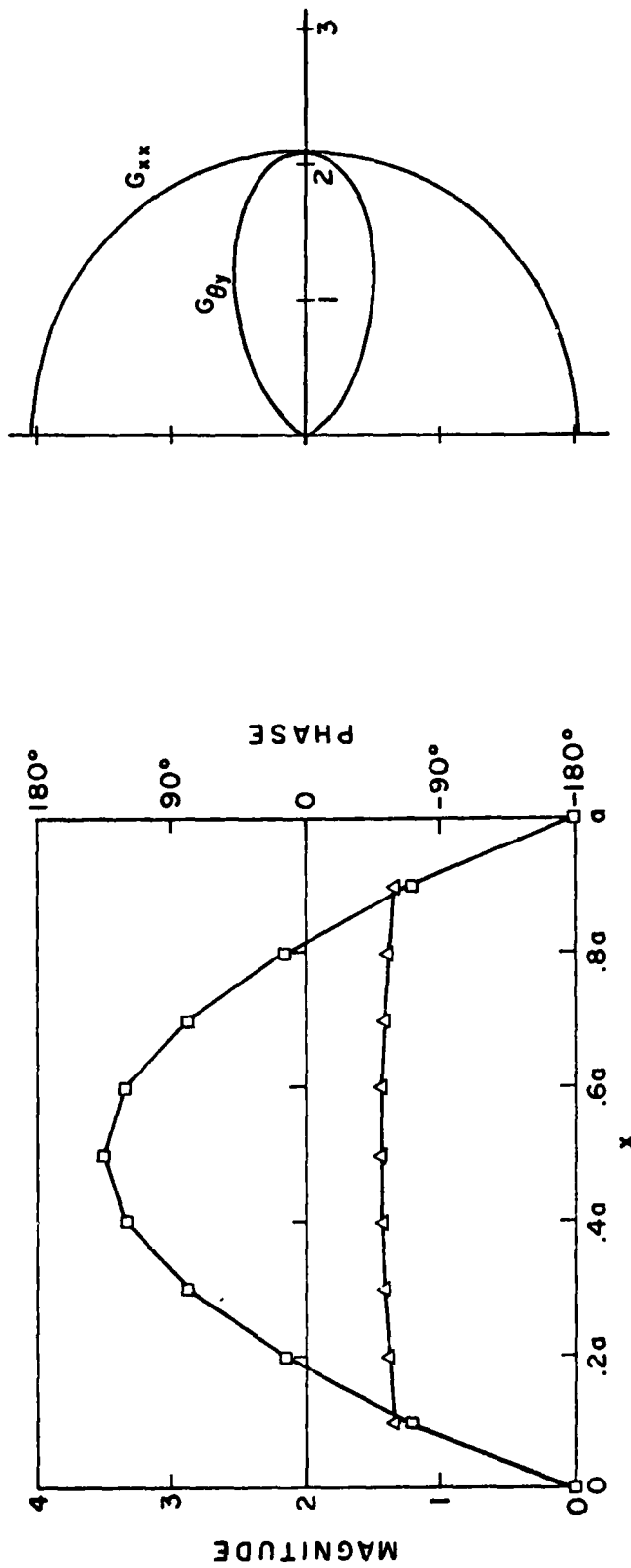
(a) Magnetic current

(b) Gain patterns

Fig. 4-3. Equivalent magnetic current M_x and radiation gain patterns for a rectangular waveguide of dimensions λ by $\lambda/2$ radiating through a centered rectangular slot of dimensions λ by $\lambda/10$. Squares denote magnitude and triangles denote phase.

of the incident wave. The phase of $\underline{M}$ is with respect to that of this $\underline{E}$ field at the aperture. All computations are for dominant TE_{10} mode excitation. Figure 4-3(b) shows the radiation gain patterns in the two planes x = 0 and y = 0. The notation $G_{\theta y}$ denotes the gain pattern due to H_θ in the y = 0 plane. The notation G_{xx} denotes the gain pattern due to H_x in the x = 0 plane. The horizontal axis in Fig. 4-3(b) is the z axis.

Figure 4-4 shows a plot of the equivalent aperture admittance Y_{ap} seen by the dominant mode of an open-ended square waveguide of width a radiating into half space. It is defined by (4-23), where
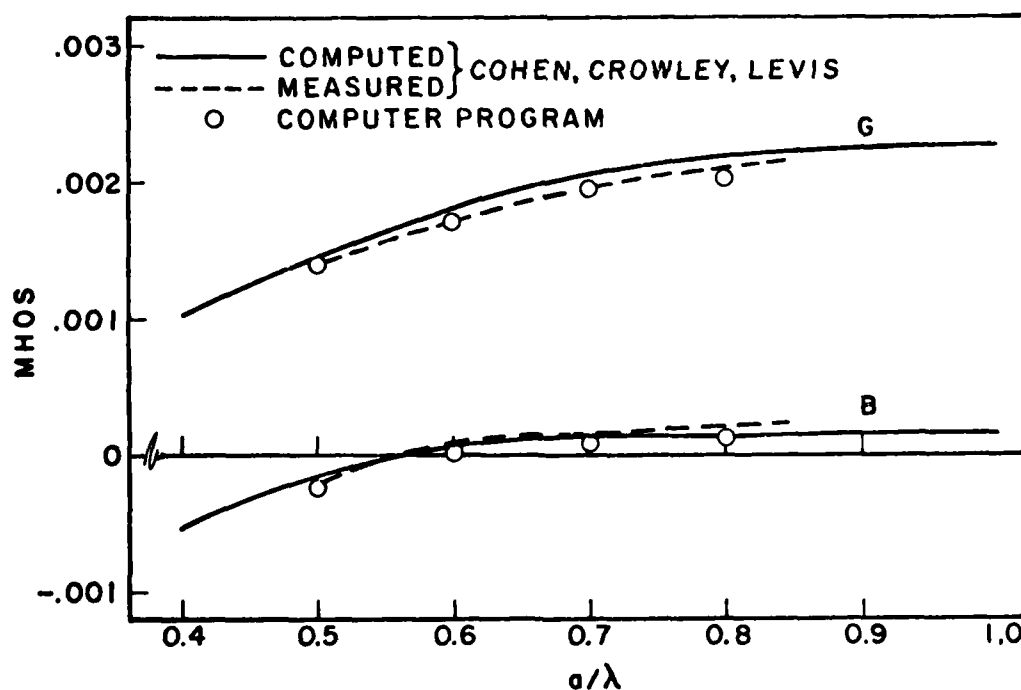
Fig. 4-4. The equivalent aperture admittance seen by the dominant mode for an open-ended square waveguide of width a radiating into half space. Our computed results are compared to those calculated and measured by Cohen, Crowley, and Levis [3].

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Γ_o is the reflection coefficient and Y_o is the characteristic wave

impedance, both for the dominant mode. Our computations are com-

pared to some previously obtained by Cohen, Crowley, and Levis [3].

Also shown are measured values reported in [3]. Additional numerical

computations are given in the report [2].

4-4. REFERENCES FOR CHAPTER IV

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Chapter V

CAVITY-BACKED APERTURES

5-1 THEORY FOR A CYLINDRICAL CAVITY

An important special case of the cavity-backed aperture is that
where the cavity is a finite cylinder of arbitrary cross section. The
aperture exists at one end of the cylinder, and the other end is com-
pletely covered by a conductor. The cavity can then be viewed as a
short-circuited waveguide and waveguide theory applied. Figure 5-1
represents a typical problem, where the excitation is by an incident
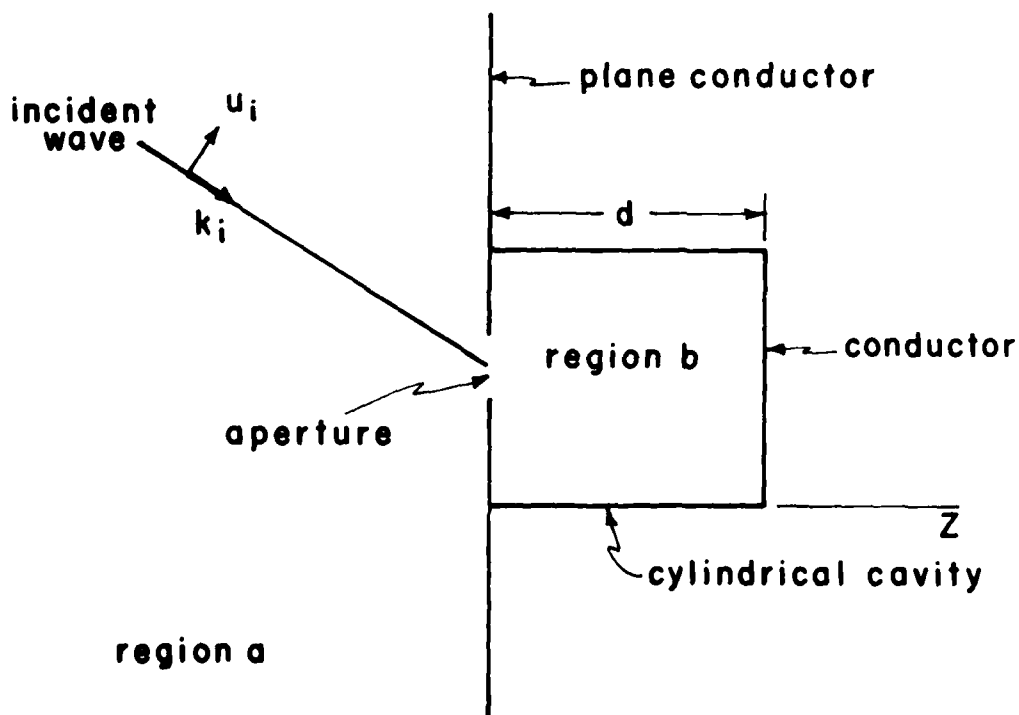plane wave from the half-space region.

Fig. 5-1. A cavity-backed aperture where the cavity is
formed by a shorted waveguide.

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The half-space part of the problem is identical to that treated in Chapter III. For a unit incident plane wave of polarization $\underline{u}_i$ and propagation vector $\underline{k}_i$, the excitation vector remains that of (3-5). For a given set of expansion and testing functions, the aperture admittance matrix of region a remains the same as that for Section 3-1, denoted by $[Y^{hs}]$. The new aspects of the problem are those of determining the aperture admittance matrix for the cavity region, which we denote by $[Y^{cav}]$, and the measurement vectors $\vec{I}^m$ for various desired field quantities.

The field in the cavity region can be expanded in terms of short-circuited waveguide modes. For a cavity depth d, the transverse to z components of an arbitrary field can be written as

$$\underline{E}_t = - \sum_i A_i \, \underline{e}_i \, \frac{\sin k_i(d-z)}{\sin k_i d} \tag{5-1}$$

$$\underline{H}_t = j \sum_i A_i Y_i (\underline{u}_z \times \underline{e}_i) \, \frac{\cos k_i(d-z)}{\sin k_i d} \tag{5-2}$$

Here $\underline{e}_i$ are the normalized modal electric field vectors (discussed in Chapter IV, Y_i are the modal characteristic admittances, and k_i are the modal wave numbers. The term $\sin k_i d$ is placed in the denominator for later convenience. The summations in (5-1) and (5-2) are assumed to be over all modes, both TE and TM if necessary. In particular, the modal wave numbers are

$$k_i = \begin{cases} k\sqrt{1 - (\lambda/\lambda_{ic})^2} \, , & \lambda < \lambda_{ic} \\ -jk_{ic}\sqrt{1 - (\lambda_{ic}/\lambda)^2} \, , & \lambda > \lambda_{ic} \end{cases} \tag{5-3}$$

where $\lambda_{ic} = 2\pi/k_{ic}$ is the ith mode cut-off wavelength. The modal characteristic admittances are

$$Y_i = \begin{cases} k_i/\omega\mu , & \text{TE modes} \\ \\ \omega\varepsilon/k_i , & \text{TM modes} \end{cases} \tag{5-4}$$

The intrinsic wavenumber k may be complex if μ and/or ε are complex to account for a lossy medium in the cavity. The $\underline{e}_i$ are normalized modal vectors, so the modal orthogonality relationships are

$$\iint\limits_S \underline{e}_i \cdot \underline{e}_j \, ds = \begin{cases} 0 , & i \neq j \\ \\ 1 , & i = j \end{cases} \tag{5-5}$$

Here the surface of integration S is the cavity (waveguide) cross section.

To evaluate the aperture admittance (2-10) in the cavity region, we consider a single expansion function $\underline{M}_n$ on the $z = 0$ plane inside the cavity. The only source in the cavity region is $\underline{M}_n$, hence the tangential field produced by $\underline{M}_n$ is of the form of (5-1) and (5-2). Using the boundary condition $\underline{M}_n = -\underline{u}_z \times \underline{E}$ and specializing (5-1) to $z = 0$, we have

$$\underline{M}_n = -\underline{u}_z \times \underline{E}_t \Big|_{z=0} = \sum_i A_{ni}\underline{u}_z \times \underline{e}_i \tag{5-6}$$

Here the additional subscript n is placed on A_{ni} to denote that it is that due to $\underline{M}_n$. Multiplying each side of (5-6) by $\underline{u}_z \times \underline{e}_j$ and integrating over the cross section S, we obtain

$$\iint\limits_S \underline{M}_n \cdot \underline{u}_z \times \underline{e}_j ds = \sum_i A_{ni} \iint\limits_S (\underline{u}_z \times \underline{e}_i) \cdot (\underline{u}_z \times \underline{e}_j) ds \tag{5-7}$$

By orthogonality (5-5), all terms of the summation are zero except the i=j term. Hence,

$$A_{ni} = \iint\limits_{\text{apert.}} \underline{M}_n \cdot \underline{u}_z \times \underline{e}_i \, ds \qquad (5\text{-}8)$$

We have replaced the integral over S by one over the aperture because $\underline{M}_n$ exists only in the aperture region.

The elements of the aperture admittance matrix (2-10) for the cavity region are now given by

$$Y_{mn}^{cav} = - \iint\limits_{\text{apert.}} \underline{W}_m \cdot \underline{H}_t^b(\underline{M}_n) \, ds \qquad (5\text{-}9)$$

The $\underline{H}_t^b$ of (5-9) is given by (5-2) evaluated at $z = 0$, so (5-9) becomes

$$Y_{mn}^{cav} = - j \sum_i A_{ni} Y_i \cot(k_i d) \iint\limits_{\text{apert.}} \underline{W}_m \cdot \underline{u}_z \times \underline{e}_i \, ds \qquad (5\text{-}10)$$

We now define the constants

$$B_{mi} = \iint\limits_{\text{apert.}} \underline{W}_m \cdot \underline{u}_z \times \underline{e}_i \, ds \qquad (5\text{-}11)$$

which are similar in form to A_{ni} given by (5-8). Then the admittances (5-10) are given by

$$Y_{mn}^{cav} = -j \sum_i A_{ni} B_{mi} Y_i \cot(k_i d) \qquad (5\text{-}12)$$

Hence, the elements of $[Y^{cav}]$ are linear combinations of the input waveguide admittances for each short-circuited waveguide mode.

Some specific quantities of interest in the solution are (a) the equivalent magnetic current $\underline{M}$, or tangential $\underline{E}$ in the aperture, (b) the amplitude of some specific mode, and (c) the electric field intensity at some point in the cavity. Once $\underline{M}$ is obtained, then quantities (b) and (c) are easily obtained.

We have the equivalent magnetic current given by (2-5) where V_n are the elements of $\vec{V}$, obtained from (2-14) specialized to the present problem. This result is

$$\vec{V} = [Y^{hs} + Y^{cav}]^{-1} \vec{P}^i \qquad (5-13)$$

where the elements of $\vec{P}^i$ are given by (3-5). The modal amplitudes A_i are obtained from (5-6) as it applies to the total magnetic current in the cavity, which is $-\underline{M}$. The result is

$$-\underline{M} = \sum_i A_i \underline{u}_z \times \underline{e}_i \qquad (5-14)$$

Again we multiply each side by $\underline{u}_z \times \underline{e}_j$, and integrate over the cross section S, as in (5-7), to obtain

$$A_i = \iint_{\text{apert.}} (-\underline{M}) \cdot \underline{u}_z \times \underline{e}_i ds \qquad (5-15)$$

Substituting for $\underline{M}$ from (2-5), and using (5-13), we have

$$A_i = - \tilde{A}\vec{V} = -\tilde{A}[Y^{hs} + Y^{cav}]^{-1} \vec{P}^i \qquad (5-16)$$

where $\tilde{A}$ is the row vector with elements given by (5-8). Hence, the measurement vector for obtaining mode amplitudes is the vector $\vec{A}$.

Finally, to obtain the $\underline{E}_t$ field at any point in the cavity, we use (5-1) with the A_i given by (5-16). At a frequency near resonance, that is, where $\sin k_j d \approx 0$ for some j, a good approximation to the field (5-1) is given by only the jth term of (5-1), or

$$\underline{E}_t \approx - A_j \underline{e}_j \frac{\sin k_j (d-z)}{\sin k_j d} \qquad (5-1.)$$

If the cavity is truly loss free, then (5-17) seems to predict an infinite field at resonance. This, however, is not correct, since $A_j \to 0$ as $\sin k_j d \to 0$ giving a finite value for $\underline{E}_t$ at resonance. The excitation of the resonant mode must then be determined from the orthogonality condition

$$\iint\limits_{apert.} \underline{M} \cdot \underline{u}_z \times \underline{e}_j \; ds = 0 \tag{5-18}$$

However, the theory for this special case will not be considered here, since it requires considerable modification of the formulas of this section.

5-2. APERTURE BACKED BY AN ARBITRARY CAVITY

When the cavity backing the aperture is of arbitrary shape, there are several ways of calculating the field in the cavity. One way is completely modal. This is the method used in this chapter. Another way is a nonmodal approach, similar to that used in [2]. That method, however, will not be considered here.

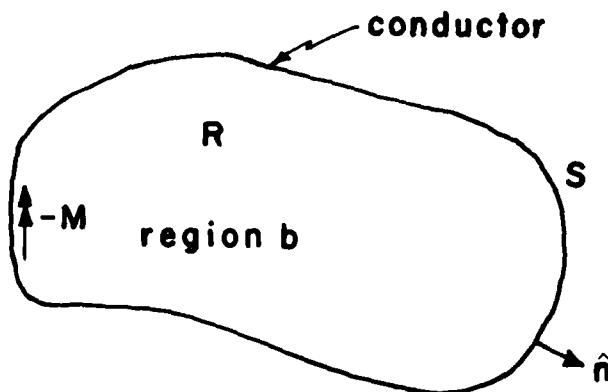Figure 5-2 represents the cavity part of a problem of the type of

Fig. 5-2. Equivalence for cavities of arbitrary shape.

Fig. 5-1, except that the cavity is now of arbitrary shape. It is desired to calculate the field $\underline{E}$ and $\underline{H}$ in the cavity region R, completely enclosed by a conducting surface S, and excited by the equivalent current $\underline{M}$ over the aperture surface. In general, the magnetic field in R can be represented as the sum of a curl-free (irrotational) part $\underline{H}^{cf}$ plus a divergence-free (solenoidal) part $\underline{H}^{df}$, that is

$$\underline{H} = \underline{H}^{cf} + \underline{H}^{df} \tag{5-19}$$

The division (5-19) is not necessarily unique, since part of the field may be both curl-free and divergence-free. For simplicity, we restrict consideration to simply-bounded and simply-connected cavities.

The modal representation of the magnetic field in a cavity bounded by a perfect electric conductor is available in textbooks [3], [4]. The use of this representation for aperture problems is summarized well in [5]. The divergence-free part of the field can be represented in terms of resonant cavity modes $\underline{H}_i$ as

$$\underline{H}^{df} = j\omega\epsilon \sum_i \frac{\underline{H}_i(\underline{r})}{k^2-k_i^2} \iint_{apert.} (-\underline{M}) \cdot \underline{H}_i \, ds \tag{5-20}$$

The $\underline{H}_i$ are solutions to the eigenvalue problem

$$\underline{\nabla} \times \underline{\nabla} \times \underline{H}_i = k_i^2\underline{H}_i \quad \text{in R} \tag{5-21}$$

$$\underline{n} \times \underline{\nabla} \times \underline{H}_i = 0 \quad \text{on S} \tag{5-22}$$

The eigenvalues k_i^2 are real, and only $k_i^2 > 0$ are used in (5-20). The eigenvectors $\underline{H}_i$ are also real, and orthonormal according to

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$$\iiint\limits_{R} \underline{H}_i \cdot \underline{H}_j \, ds = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad (5\text{-}23)$$

The $\underline{H}_i$ defined above are the usual resonant modes of time-harmonic

fields in the cavity.

The curl-free part of the field can be expressed in terms of

"static modes" as

$$\underline{H}^{cf} = \frac{-1}{j\omega\mu} \sum_i \underline{G}_i(\underline{r}) \iint\limits_{apert.} (-\underline{M}) \cdot \underline{G}_i \, ds \qquad (5\text{-}24)$$

where

$$\underline{G}_i = -\nabla\psi_i \qquad (5\text{-}25)$$

$$-\nabla^2\psi_i = \nu_i^2\psi_i \qquad \text{in} \quad R \qquad (5\text{-}26)$$

$$\frac{\partial\psi_i}{\partial n} = 0 \qquad \text{on} \quad S \qquad (5\text{-}27)$$

The modal vectors $\underline{G}_i$ satisfy (5-21) and (5-22) for $k_i^2 = 0$, that is,

for zero eigenvalue. Hence, we can think of $k_i^2 = 0$ as being an eigen-

value of infinite degeneracy. The $\underline{G}_i$ are real and orthonormal accord-

ing to

$$\iiint\limits_{R} \underline{G}_i \cdot \underline{G}_j \, ds = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad (5\text{-}28)$$

Also, since $\underline{H}^{df}$ is orthogonal to $\underline{H}^{cf}$ in general, we have

$$\iiint\limits_{R} \underline{G}_i \cdot \underline{H}_j \, ds = 0 \qquad (5\text{-}29)$$

for all i and j. The total field in R is simply the sum of (5-20)

and (5-24) according to (5-19).

There is an alternative form for (5-24) which is sometimes convenient. This is

$$\underline{H}^{cf} = - \underline{\nabla}\psi \tag{5-30}$$

where

$$\psi = \frac{1}{\mu} \sum_i \psi_i(\underline{r}) \iint_{\text{apert.}} (-m)\psi_i \, ds \tag{5-31}$$

Here m is the magnetic charge associated with $\underline{M}$ according to the equation of continuity

$$m = \frac{-1}{j\omega} \underline{\nabla} \cdot \underline{M} \tag{5-32}$$

The ψ_i are still solutions to the eigenvalue problem (5-26) and (5-27), and orthonormal according to (5-28) where $\underline{G}_i$ is given by (5-25). The derivation of (5-31) from (5-24) involves substitution for $\underline{G}_i$ from (5-25) and application of the divergence theorem to the integral over R.

As shorthand notation, we can define the set of modes $\{\underline{F}_i\}$ to be $\{\underline{G}_i, \underline{H}_i\}$, that is, to include all modes. Then the total magnetic field can be written as

$$\underline{H} = j\omega\varepsilon \sum_i \frac{\underline{F}_i(\underline{r})}{k^2-k_i^2} \iint_{\text{apert.}} (-\underline{M}) \cdot \underline{F}_i \, ds \tag{5-33}$$

where

$$\iiint_R \underline{F}_i \cdot \underline{F}_j \, ds = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \tag{5-34}$$

This is the form used in Sec. 8-13 of [1]. However, we must remember that there are an infinite set of modes associated with the eigenvalue $k_i^2 = 0$.

To obtain the aperture admittance elements for the cavity region we use (5-9) where $\underline{H}^b$ is now given by (5-33) with $\underline{M}$ replaced by $\underline{M}_n$, or

the sum of (5-20) and (5-24) with $\underline{M}$ replaced by $\underline{M}_n$. The result is

$$Y_{mn}^{cav} = \frac{1}{j\omega\mu} \sum_i a'_{mi} a_{ni} + j\omega\varepsilon \sum_i \frac{b'_{mi} b_{ni}}{k_i^2 - k^2} \qquad (5\text{-}35)$$

where

$$\begin{bmatrix} a_{ni} \\ \\ a'_{ni} \end{bmatrix} = \iint_{apert.} \begin{bmatrix} \underline{M}_n \\ \\ \underline{W}_n \end{bmatrix} \cdot \underline{G}_i \, ds \qquad (5\text{-}36)$$

$$\begin{bmatrix} b_{ni} \\ \\ b'_{ni} \end{bmatrix} = \iint_{apert.} \begin{bmatrix} \underline{M}_n \\ \\ \underline{W}_n \end{bmatrix} \cdot \underline{H}_i \, ds \qquad (5\text{-}37)$$

Once again we have the possibility that, for a loss-free cavity, one or more terms of (5-35) can become infinite at resonance, that is, when $k = k_j$. In the case of lossy cavities this does not occur, although numerical problems may arise if the cavity is only slightly lossy.

The magnetic field produced by the nth expansion function is given by substituting $\underline{M}_n$ into the sum of (5-20) and (5-24), or

$$\underline{H}(\underline{M}_n) = \frac{-1}{j\omega\mu} \sum_i a_{ni}\underline{G}_i + j\omega\varepsilon \sum_i \frac{b_{ni} \underline{H}_i}{k^2 - k_i^2} \qquad (5\text{-}38)$$

where a_{ni} and b_{ni} are given by (5-36) and (5-37). The electric field is obtained from (5-38) according to $j\omega\varepsilon\underline{E} = \underline{\nabla} \times \underline{H}$, resulting in

$$\underline{E}(\underline{M}_n) = \sum_i \frac{b_{ni} \underline{\nabla} \times \underline{H}_i}{k^2 - k_i^2} \qquad (5\text{-}39)$$

Note that the first summation of (5-38) vanishes when (5-39) is derived, since $\underline{\nabla} \times \underline{G}_i = 0$. Both $\underline{E}$ and $\underline{H}$ are linearly related to

magnetic current. Remembering that the source in region b of the original problem is $-\underline{M}$, we can use the superposition (2-5) to obtain the total magnetic and electric fields in the cavity as

$$\underline{H} = - \sum_n V_n \underline{H}(\underline{M}_n) \tag{5-40}$$

$$\underline{E} = - \sum_n V_n \underline{E}(\underline{M}_n) \tag{5-41}$$

Here the elements of $\vec{V}$ are obtained from (2-14) in general. For the particular case of a cavity-backed aperture in a conducting plane, $\vec{V}$ is given by (5-13) where $[Y^{cav}]$ now has the elements (5-35). In (5-40), the coefficient of the jth static mode $\underline{G}_j$ is

$$a_j = \frac{1}{j\omega\mu} \sum_n V_n a_{nj} = \frac{1}{j\omega\mu} \tilde{a}_j \vec{V} \tag{5-42}$$

where $\tilde{a}_j$ is the row vector with elements a_{nj} given by (5-36). Using (5-13) for $\vec{V}$, we have for the amplitude of the jth static mode

$$a_j = \frac{1}{j\omega\mu} \tilde{a}_j [Y^{hs} + Y^{cav}]^{-1} \vec{p}^i \tag{5-43}$$

Similarly, the coefficient of the jth resonator mode $\underline{H}_j$ in (5-40) is

$$b_j = \frac{-j\omega\varepsilon}{k^2-k_j^2} \sum_n V_n b_{nj} = \frac{-j\omega\varepsilon}{k^2-k_j^2} \tilde{b}_j \vec{V} \tag{5-44}$$

where $\tilde{b}_j$ is the row vector with elements b_{nj} given by (5-37). Again using (5-13) for $\vec{V}$, we have for the amplitude of the jth resonator mode

$$b_j = \frac{-j\omega\varepsilon}{k^2-k_j^2} \tilde{b}_j [Y^{hs} + Y^{cav}]^{-1} \vec{p}^i \tag{5-45}$$

Hence, $\vec{a}_j/j\omega\mu$ and $-j\omega\varepsilon\vec{b}_j/(k^2-k_j^2)$ are the measurement vectors for determining a mode amplitude.

In the vicinity of a resonant frequency in a relatively loss-free cavity, only one term of (5-38) and (5-39) may suffice to approximate the field. For example, if k is near k_j, then

$$\underline{H} \approx b_j \, \underline{H}_j \tag{5-46}$$

$$\underline{E} \approx \frac{b_j \, \nabla \times \underline{H}_j}{j\omega\varepsilon} \tag{5-47}$$

If the cavity is truly loss-free, then (5-47) appears to predict an infinite field at resonance. Again this is not correct, since $\sum_n V_n b_{nj} \to 0$ as $k \to k_j$. In this case the excitation of the resonant mode must be determined from the orthogonality condition

$$\iint\limits_{\text{apert.}} \underline{M} \cdot \underline{H}_j \, ds = 0 \tag{5-48}$$

Again we will not consider this special case here, since considerable modification of the formulation is required.

5-3 DISCUSSION

A general formulation for a cavity-backed aperture in an infinite conducting plane has been given in terms of generalized network parameters. The approach is also valid for a cavity-backed aperture in a conducting body of arbitrary shape, but then the calculation of the excitation and/or the measurement vectors is more difficult.

Two interesting specializations of the general theory are (a) the low frequency case where cavity dimensions are small compared to wavelength

and (b) the resonant case, particularly in the vicinity of the first resonance. It is hoped that the general theory can be simplified for these cases, allowing us to derive relatively simple equivalent circuits. These topics require further investigation.

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Chapter VI

ELECTROMAGNETIC TRANSMISSION THROUGH NARROW SLOTS

IN THICK CONDUCTING SCREENS

6-1 INTRODUCTION

The first accurate treatment of electromagnetic coupling through

small holes was for zero-thickness conductors, and used the concept of

aperture polarizability [1], [2]. This theory has found extensive appli-

cation in the literature (see [3] for a bibliography). When the aperture

becomes larger and/or the conductor has thickness, the polarizability

concept becomes inadequate and the generalized admittance concept should

be used. In this chapter we apply the admittance concept to narrow,

infinitely long slots (sometimes called slits) in a conducting plane of

finite thickness. The concepts we use are general, applying also to

three-dimensional problems, but we here consider only the two-dimensional

case.

The problem of a slot in a thick conducting screen has been con-

sidered by several methods (see [4] for a bibliography). It was treated

by the generalized admittance concept for the TE (transverse electric

to the slot axis) case in [4]. A similar solution for the TM (transverse

magnetic to the slot axis) case can be found in [5]. For narrow slots

in thick conductors, only the transmission line mode, which is a TE mode,

can propagate through the slot region. All higher order TE modes, and

all TM modes, are cut off. Hence, we restrict explicit consideration

to the TE case. The solution used is basically a simplification of the

general solution [4] for the narrow slot. This results in relatively

simple formulas, and a simple equivalent circuit, for narrow slots in conducting screens of finite thickness.

6-2 FORMULATION OF THE PROBLEM

The problem to be considered is shown in Fig. 6-1, which represents the cross section of a conducting screen of thickness d in which a slot
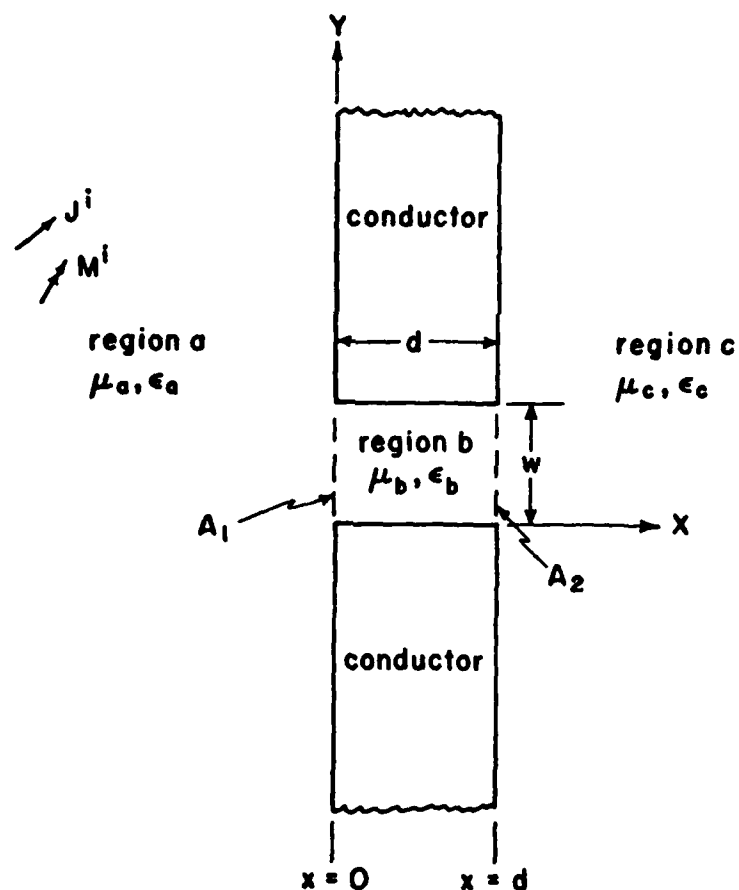

Fig. 6-1. A uniform slot of width w in a perfectly conducting screen of thickness d.

of uniform width w is cut. The left-hand half space (x < 0) is called

region a, the uniform slot region (0 < x < d, 0 < y < w) is called

region b, and the right-hand half space (x > d) is called region c. The

boundary common to regions a and b is called the aperture A_1. The

boundary common to regions b and c is called the aperture A_2. Regions

a, b, and c are each filled with homogeneous media of constitutive

parameters (μ_a, ε_a), (μ_b, ε_b), and (μ_c, ε_c) respectively. Each μ and ε

can be considered complex to account for dissipation. The excitation

is due to known sources $\underline{J}^i$ and $\underline{M}^i$ in region a. It is desired to obtain

a solution for the field in each region, and for the power transmitted

into region c.

As described in Chapter II, the equivalence principle can be used to

divide the original problem into three equivalent parts, as shown in

Fig. 6-2. In Fig. 6-2a, we have the original sources $\underline{J}^i$, $\underline{M}^i$, plus the

equivalent magnetic current $\underline{M}_1$, where

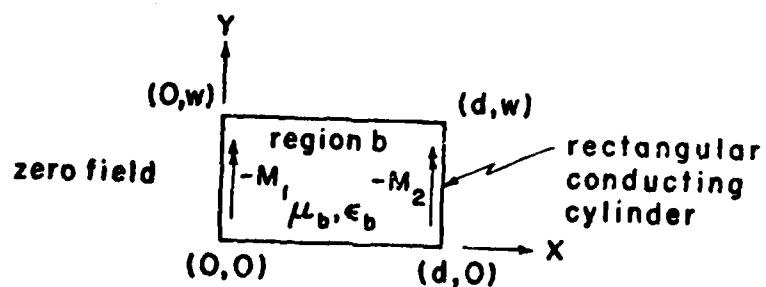$$\underline{M}_1 = \hat{\underline{x}} \times \underline{E} \qquad\qquad (6\text{-}1)$$

over the aperture region A_1, all radiating in the presence of a complete

conductor (aperture A_1 shorted). In (6-1), $\hat{\underline{x}}$ is the x-directed unit vector,

normal to A_1, and $\underline{E}$ is the electric field in the aperture A_1 in the original

problem. In Fig. 6-2b, we have the equivalent magnetic currents $-\underline{M}_1$, given

by (6-1), over A_1, and $-\underline{M}_2$, given by

$$\underline{M}_2 = -\hat{\underline{x}} \times \underline{E} \qquad\qquad (6\text{-}2)$$
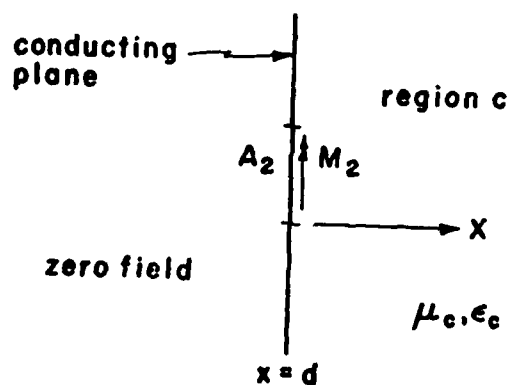
over the aperture region A_2, all radiating in the presence of a conductor

completely enclosing the rectangular region b (both apertures shorted).
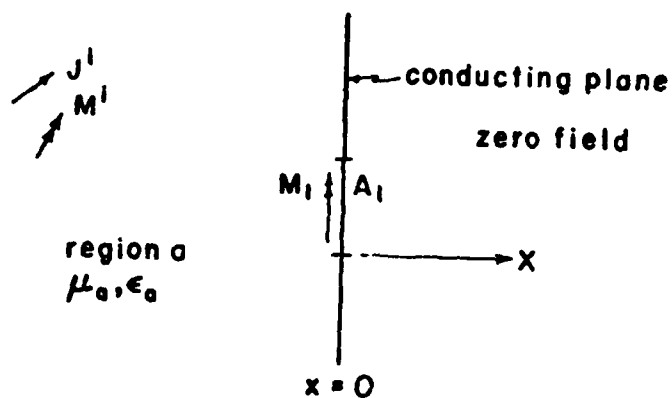
(a) Equivalence for region a

(b) Equivalence for region b.

(c) Equivalence for region c.

Fig. 6-2. The problem divided into three equivalent parts.

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In (6-2), the unit vector $\hat{\underline{x}}$ is normal to A_2, and $\underline{E}$ is the electric field in the aperture A_2 in the original problem. Finally, in Fig. 6-2c, we have the equivalent magnetic current $\underline{M}_2$, given by (6-2), over the aperture region A_2, radiating in the presence of a complete conducting plane (aperture A_2 shorted).

The use of $\underline{M}_1$ in Fig. 6-2a and $-\underline{M}_1$ in Fig. 6-2b ensures the continuity of the tangential components of $\underline{E}$ across the aperture A_1. The use of $-\underline{M}_2$ in Fig. 6-2b and $\underline{M}_2$ in Fig. 6-2c ensures the continuity of the tangential components of $\underline{E}$ across the aperture A_2. The remaining boundary conditions to be enforced are the continuity of tangential components of $\underline{H}$ across A_1 and A_2. By the method used in [4], we obtain the following two operator equations:

$$-\underline{H}_t^a(\underline{M}_1) - \underline{H}_t^b(\underline{M}_1) - \underline{H}_t^b(\underline{M}_2) = \underline{H}_t^{sc} \qquad \text{over } A_1 \qquad (6\text{-}3)$$

$$-\underline{H}_t^b(\underline{M}_1) - \underline{H}_t^b(\underline{M}_2) - \underline{H}_t^c(\underline{M}_2) = 0 \qquad \text{over } A_2 \qquad (6\text{-}4)$$

Here $\underline{H}_t^p(\underline{M}_q)$ is the operator which gives the tangential component of $\underline{H}$ due to the current $\underline{M}_q$ radiating in region p, with all apertures shorted. $\underline{H}_t^{sc}$ is the tangential component of $\underline{H}$ due to the impressed sources $\underline{J}^i$, $\underline{M}^i$ in region a with aperture A_1 shorted. In terms of the current $\underline{J}^{sc}$ which exists on the shorted aperture A_1 in Fig. 6-2a,

$$\underline{H}_t^{sc} = \hat{\underline{x}} \times \underline{J}^{sc} \qquad (6\text{-}5)$$

In Chapter II, $\underline{H}^{sc}$ was called the field due to the impressed sources and denoted by $\underline{H}^i$. Others [6] have called it the generator field and denoted it by $\underline{H}_g$.

A convenient way to reduce the operator equations (6-3) and (6-4) to matrix equations is the method of moments [7]. The procedure can be summarized as follows. Define sets of expansion functions $\{\underline{M}_{1n}\}$ in A_1 and $\{\underline{M}_{2n}\}$ in A_2, and express the equivalent magnetic currents as

$$\underline{M}_1 = \sum_{n=1}^{N_1} V_{1n} \underline{M}_{1n} \tag{6-6}$$

$$\underline{M}_2 = \sum_{n=1}^{N_2} V_{2n} \underline{M}_{2n} \tag{6-7}$$

where V_{1n} and V_{2n} are coefficients to be determined. Define a symmetric product for each slot as

$$\langle \underline{A}, \underline{B} \rangle_q = \int_0^w \underline{A} \cdot \underline{B} \, dy \tag{6-8}$$

where $q = 1$ or 2. The integrand in (6-8) is evaluated at $x = 0$ when $q = 1$ and at $x = d$ when $q = 2$. Define sets of testing functions $\{\underline{W}_{1m}\}$ in A_1 and $\{\underline{W}_{2m}\}$ in A_2. Substitute (6-6) and (6-7) into (6-3) and (6-4), and test the resultant equations with each $\underline{W}_{1m}$, $m=1,2,\ldots,N_1$, and $\underline{W}_{2m}$, $m=1,2,\ldots,N_2$. The result is

$$[Y_{11}^a]\vec{V}_1 + [Y_{11}^b]\vec{V}_1 + [Y_{12}^b]\vec{V}_2 = \vec{I}^i \tag{6-9}$$

$$[Y_{21}^b]\vec{V}_1 + [Y_{22}^b]\vec{V}_2 + [Y_{22}^c]\vec{V}_2 = \vec{0} \tag{6-10}$$

where

$$[Y_{qr}^p] = [- \langle W_{qm}, H_t^p(M_{rn}) \rangle_q]_{N_q \times N_r} \tag{6-11}$$

$$\vec{V}_r = [V_{rn}]_{N_r \times 1} \tag{6-12}$$

$$\vec{I}^i = [\langle W_{1m}, H_t^{sc} \rangle_1]_{N_1 \times 1} \tag{6-13}$$

The matrices $[Y_{qr}^p]$ are called the generalized admittances, the vectors $\vec{V}_r$ are called the generalized voltages, and the vector $\vec{I}^i$ is called the generalized source current. A solution to the problem is obtained by solving the matrix equations (6-9) and (6-10) for $\vec{V}_1$ and $\vec{V}_2$, which determine the magnetic currents by (6-6) and (6-7). Once the equivalent magnetic currents are known, the fields in each region can be obtained from the equivalent problems of Fig. 6-2.

6-3   SPECIALIZATION TO NARROW SLOTS

As noted in the introduction, rigorous solutions to the general problem can be found in the literature. We here consider only an approximation which, as we shall see, gives highly accurate solutions for narrow slots. We consider only the case of TE excitation because, as noted earlier, all TM modes in a narrow slot region are cut off.

The approximate solution is basically a one-term moment solution to the general problem. For a testing function in each aperture, we choose the constant (integrated value unity)

$$\underline{W}_{11} = \underline{W}_{21} = \underline{W} = \hat{z} \frac{1}{w} \tag{6-14}$$

This is the $\underline{H}$ field variation of the transmission line mode in region b, and is orthogonal to all higher-order waveguide modes. For now, the expansion function in each aperture

$$\underline{M}_{11} = \underline{M}_{21} = \underline{M} = \hat{z} \, f(y) \tag{6-15}$$

will be left undefined as to its functional form $f(y)$. However, so that $\underline{M}$ excites the transmission line mode the same regardless of its functional form, we require

$$\int_0^W f(y) \ dy = 1 \tag{6-16}$$

for all "trial functions" $f(y)$. In other words, the net magnetic current is unity.

The general network equations (6-9) and (6-10) now reduce to scalar equations

$$Y^a V_1 + Y_{11}^b V_1 + Y_{12}^b V_2 = I^i \tag{6-17}$$

$$Y_{21}^b V_1 + Y_{22}^b V_2 + Y^c V_2 = 0 \tag{6-18}$$

where

$$Y^a = \frac{-1}{w} \int_0^W \hat{\underline{z}} \cdot \underline{H}_t^a(\underline{M}) \ dy \tag{6-19}$$

is the aperture admittance of A_1 looking into region a, and

$$Y^c = \frac{-1}{w} \int_0^W \hat{\underline{z}} \cdot \underline{H}_t^c(\underline{M}) \ dy \tag{6-20}$$

is the aperture admittance of A_2 looking into region c. The matrix of admittances

$$\begin{bmatrix} Y_{11}^b & Y_{12}^b \\ Y_{21}^b & Y_{22}^b \end{bmatrix} = \begin{bmatrix} -jY_o \cot k_b d & -jY_o \csc k_b d \\ -jY_o \csc k_b d & -jY_o \cot k_b d \end{bmatrix} \tag{6-21}$$

is the two-port admittance matrix for a parallel-plate transmission line of length d, characteristic admittance $Y_o = 1/w\eta_b$, and propagation constant jk_b, where η_b and k_b are the intrinsic impedance and wave number of region b. The impressed current is

$$I^i = \frac{1}{w} \int_0^w H_z^{sc} \, dy \qquad (6-22)$$

which is the average surface density of electric current over the

short-circuited aperture. The approximations to the equivalent

magnetic currents in the apertures A_1 and A_2 are now

$$\underline{M}_1 = V_1 \underline{M} \quad \text{and} \quad \underline{M}_2 = V_2 \underline{M} \qquad (6-23)$$

respectively. The equivalent circuit for this approximation to the narrow

slot is shown in Fig. 6-3.

It is well-known that the aperture admittance of a thin slot open-

ing into half space is insensitive to small variations in the tangential

electric field in the slot about its true value [8]. We here consider

three cases, all of which result in an aperture admittance of the form

$$Y^a = \frac{1}{\eta_a \lambda_a} [\pi - 2j \ln(C \, k_a w)] \qquad (6-24)$$

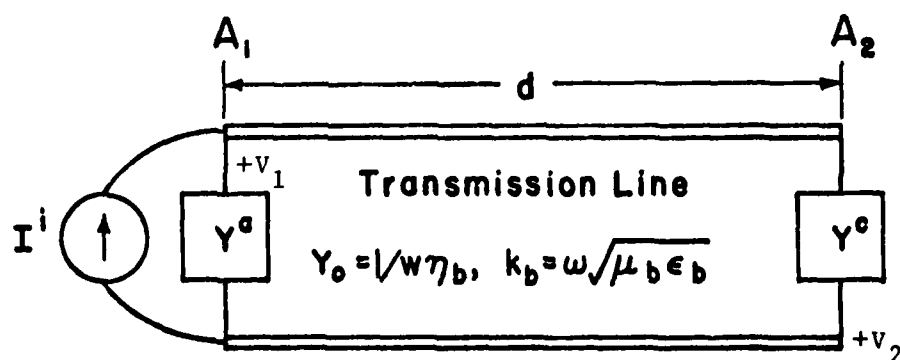for region a, and of the same form with all a's replaced by c's for

Fig. 6-3. Equivalent circuit for a narrow slot in a thick
conducting screen.

region c. In (6-24), η_a, λ_a, and k_a are the intrinsic impedance, wave-length, and wave number of medium a, respectively, and C is a constant depending on the choice of f(y). If the tangential electric field in the slot is assumed constant, then [8]

$$C = \gamma/2e^{3/2} \approx 0.1987 \qquad (6-25)$$

Here γ = 1.781... and e = 2.718... . If the tangential electric field is assumed to be the quasi-static solution for a right-angle flange, then [9]

$$C = \gamma/\pi e \approx 0.2086 \qquad (6-26)$$

Finally, if the tangential electric field is assumed to be the quasi-static solution for a slot in a zero-thickness screen, then [10]

$$C = \gamma/8 \approx 0.2226 \qquad (6-27)$$

Since C is in a logarithmic term in (6-24), it makes little difference which value is chosen. However, so that our solution remains strictly valid as the thickness d → 0, we choose (6-27).

A parameter of interest is the transfer admittance

$$Y_{12} = \frac{I^1}{V_2} \qquad (6-28)$$

which allows one to calculate the strength of $\underline{M}_2 = V_2 \underline{M}$ given the excitation I^1 of (6-22). From transmission line theory and circuit theory applied to the equivalent circuit of Fig. 6-3, we obtain

$$Y_{12} = -(Y^a + Y^c)\cos k_b d - j(Y_o + \frac{Y^a Y^c}{Y_o})\sin k_b d \qquad (6-29)$$

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The power transmitted through the aperture is equal to that dissipated in Y^c of the equivalent circuit, that is

$$P_{trans} = |V_2|^2 \, Re(Y^c) \tag{6-30}$$

In terms of the transfer admittance, this is

$$P_{trans} = \left| \frac{I^i}{Y_{12}} \right|^2 \, Re(Y^c) \tag{6-31}$$

where I^i is given by (6-22), Y_{12} by (6-29), and Y^c by (6-24) with a's replaced by c's.

Now consider the case of a normally incident plane wave

$$H_z^{io} = H_o \, e^{-jk_a x} \tag{6-32}$$

Here the superscript io denotes a wave in media a of infinite extent (no conductor). The power incident on the aperture (per unit length in the z direction) is then

$$P_{inc} = \eta_a |H_o|^2 \, w \tag{6-33}$$

The short-circuit magnetic field is twice the incident field on the conductor, or

$$H_z^{sc} = 2H_o \tag{6-34}$$

Hence, the impressed current I^i is, by (6-22),

$$I^i = 2H_o \tag{6-35}$$

The transmission coefficient of the slot is defined as

$$T = \frac{P_{trans}}{P_{inc}} \tag{6-36}$$

Now, substituting from (6-31), (6-33), and (6-35) into (6-36), we have

$$T = \frac{4 \ \mathrm{Re}(Y^c)}{w \eta_a |Y_{12}|^2} \tag{6-37}$$

If the plane wave is incident at some angle θ_{inc} in the x-y plane measured from the x axis, then the left-hand side of (6-37) becomes $T \cos \theta_{inc}$.

The transmission coefficient (6-37) depends on screen thickness d only through the parameter Y_{12}. In particular, it will be maximum when $|Y_{12}|$ is minimum. Consider the case of region b lossless, and denote the aperture admittances by

$$Y^a = G^a + jB^a \quad \text{and} \quad Y^c = G^c + jB^c \tag{6-38}$$

We can now write the real and imaginary parts of Y_{12} as given by (6-29) as

$$\mathrm{Re}(Y_{12}) = -(G^a + G^c)\cos k_b d + \frac{1}{Y_o}(G^a B^c + G^c B^a)\sin k_b d \tag{6-39}$$

$$\mathrm{Im}(Y_{12}) = -(B^a + B^c)\cos k_b d - (Y_o + \frac{G^a G^c - B^a B^c}{Y_o})\sin k_b d \tag{6-40}$$

For narrow slots, we see from (6-24) that $B^a \gg G^a$ and $B^c \gg G^c$. Hence, the coefficients of the trigonometric terms in (6-40) are much larger than those in (6-39), and we can minimize $|Y_{12}|$ by setting $\mathrm{Im}(Y_{12}) = 0$. As $w \rightarrow 0$, we retain only the dominant Y_o term in the coefficient of $\sin k_b d$ in (6-40). Then $\mathrm{Im}(Y_{12}) = 0$ when

$$\tan k_b d \approx -(\frac{B^a + B^c}{Y_o}) \tag{6-41}$$

which is the condition for "slot resonance." Since $B^a + B^c \ll Y_o$, a first approximation to resonance is $k_b d \approx n\pi$, or

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$$d_{res} \approx \frac{n}{2} \lambda_b, \quad n = 1, 2, 3, \ldots \tag{6-42}$$

Here the subscript "res" denotes "at resonance." Actually, the right-hand side of (6-41) is a small negative number, hence resonance occurs when d is slightly less than an integer number of half wavelengths in region b. Assuming that $\tan k_b d$ varies linearly in the vicinity of each zero, we have

$$d_{res} \approx (n - \frac{B^a + B^c}{\pi Y_o}) \frac{\lambda_b}{2} \tag{6-43}$$

where $n = 1, 2, 3, \ldots$. This is a better approximation to the resonant thickness than is (6-42).

We next wish to obtain the transmission coefficient at resonance. Now $\cos k_b d \approx (-1)^n$ and $\sin k_b d$ is small, hence from (6-39) we have

$$\mathrm{Re}(Y_{12}) \xrightarrow[w \to 0]{} - (-1)^n (G^a + G^c) \tag{6-44}$$

At resonance, $|Y_{12}| = |\mathrm{Re}(Y_{12})|$ and $\mathrm{Re}(Y^c) = \pi/\eta_c \lambda_c$, and (6-37) reduces to

$$T_{res} \xrightarrow[w \to 0]{} \frac{4\pi}{w \eta_a \eta_c \lambda_c (G^a + G^c)^2} \tag{6-45}$$

If we introduce the parameter

$$\nu = \frac{\eta_a \lambda_a}{\eta_c \lambda_c} = \frac{\varepsilon_c}{\varepsilon_a} \tag{6-46}$$

and substitute for G^a and G^c as obtained from (6-24), then (6-45) can be written as

$$T_{res} \xrightarrow[w \to 0]{} \frac{4 \lambda_a \nu}{w \pi (1 + \nu)^2} \tag{6-47}$$

If medium a and medium c are the same, $\nu = 1$ and (6-47) reduces to

$$(Tw)_{res} \xrightarrow[w \to 0]{} \frac{1}{\pi} \lambda_a \qquad (6\text{-}48)$$

The quantity Tw is the transmission width, or apparent width, of the slot. The transmission width times the incident power density equals the power transmitted by a unit length of the slot. Hence, when medium a and medium c are the same, at resonance the transmission width of a narrow slot is $1/\pi$ wavelengths, regardless of its actual width. When medium a and medium c are different, the transmission width of a narrow slow at resonance can be obtained from (6-47), where ν is given by (6-46). Note also that (6-47) and (6-48) are independent of the medium in region b, that is, the peak values of T are independent of the medium in the slot. However, the positions and widths of the peaks are a function of medium b, as is evident from (6-43) with $Y_o = 1/w\eta_b$.

6-4 NUMERICAL RESULTS

Once the transfer admittance Y_{12} is found for a given narrow slot, we can then readily compute the field transmitted into the half-space region c. It is thus desirable to compare the results for Y_{12} obtained from the equivalent circuit formula (6-29) with results for Y_{12} obtained from the higher order moment solution of [4]. In the latter, six pulse expansion functions were used on each aperture face A_1 and A_2 for all slot widths considered. These results are indicated at various points on the curves in Figs. 6-4 to 6-8 by markers (circles, triangles, pluses, and crosses). The solid lines represent results obtained directly from the equivalent circuit model and were computed a intervals of 0.01 wavelengths in region b. For narrow slots, the parameters Y_{12} and T, computed from the approximate solution, are periodic functions of d (screen thickness).

They are only plotted for $0 \leq d/\lambda_b \leq 2$, where λ_b is the wavelength in the slot region b. The permeability of all regions is that of free space and the permittivity of the different regions is specified for the different cases.

The real and imaginary parts of Y_{12}/Y_o, where $Y_o = 1/w\eta_b$ is the characteristic admittance of the parallel-plate transmission line, are shown in Fig. 6-4 for slot widths of $0.01\lambda_o$ to $0.2\lambda_o$, where λ_o is the wavelength of free space. Equation (6-28) was used for the higher order moment solution in which the voltage V_2 becomes an integral over the magnetic current $\underline{M}_2$. The two results are in excellent agreement as w becomes small. Also, as expected from (6-29) and (6-24), Y_{12} becomes purely imaginary as $w \rightarrow 0$. These computations indicate that the equivalent circuit gives highly accurate results for $w < 0.1\lambda_o$, and good results for $0.1\lambda_o < w < 0.2\lambda_o$. For $w > 0.2\lambda_o$, the results obtained from the equivalent circuit become increasingly inaccurate. This is to be expected, since the half space is then no longer adequately represented by a lumped admittance.

The transmission coefficients for the same slots are shown in Fig. 6-5, where the equivalent circuit results were obtained from (6-37). For the higher order moment solution, P_{trans} was obtained by the usual multi-port network formula [4]. As $w \rightarrow 0$, T becomes a maximum at so-called resonant thicknesses, which approach multiples of $\lambda_b/2$, as predicted by (6-42). The peak value of T for $w = 0.01\lambda_o$ in Fig. 6-5 is $T = 30.5$ occurring at $d/\lambda_b = 0.47, 0.97, 1.47, \ldots$. This is in agreement with the prediction that the transmission width $Tw \rightarrow \lambda_o/\pi$ at values of d slightly less than integral multiples of $\lambda_b/2$ as $w \rightarrow 0$.
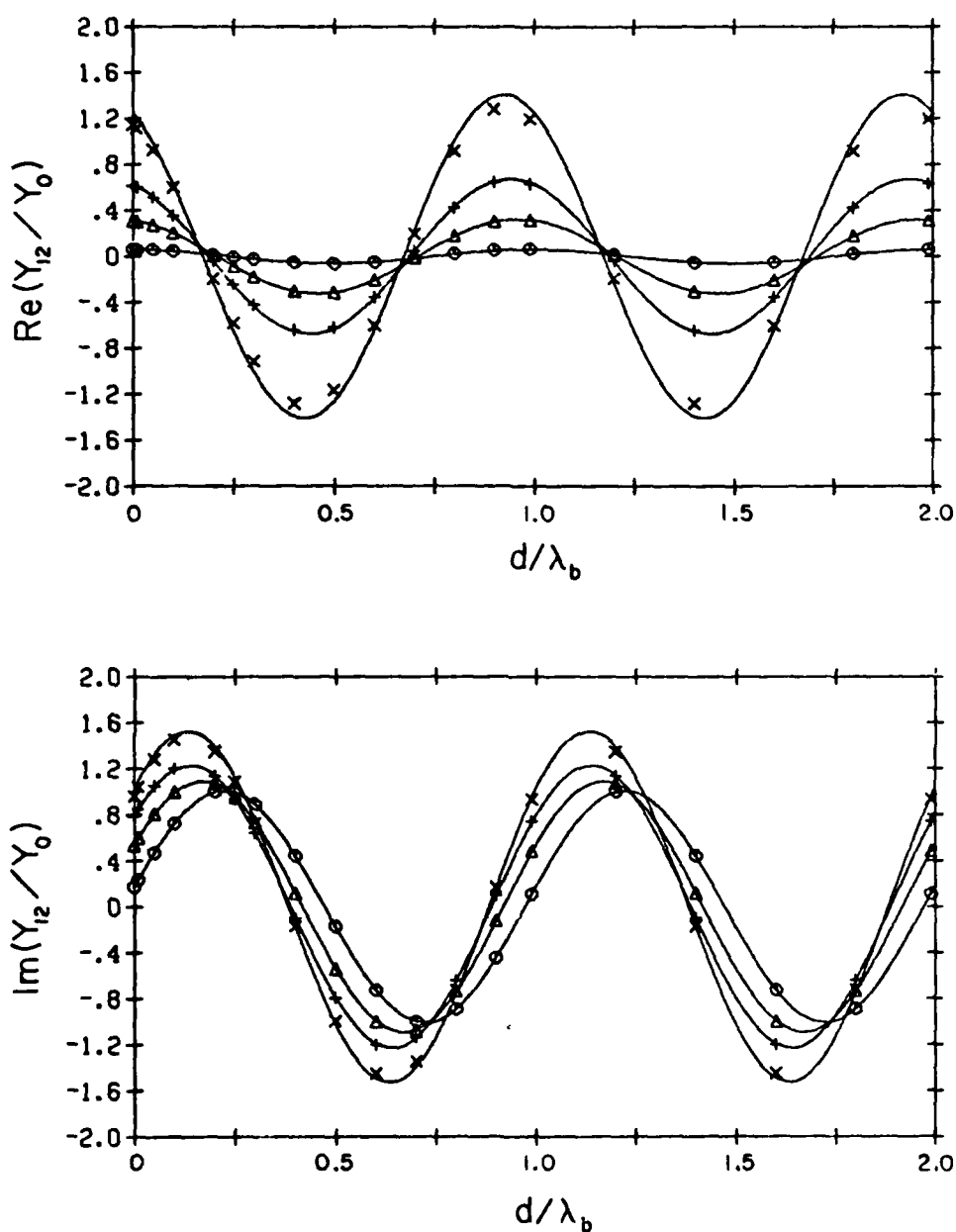
Fig. 6-4. Plots of $\text{Re}(Y_{12}/Y_o)$ and $\text{Im}(Y_{12}/Y_o)$ vs. d/λ_b for $\varepsilon_a = \varepsilon_b = \varepsilon_c$ $= \varepsilon_o$ and width $w = 0.01\lambda_o$ (circles), $w = 0.05\lambda_o$ (triangles), $w = 0.1\lambda_o$ (pluses), and $w = 0.2\lambda_o$ (crosses).
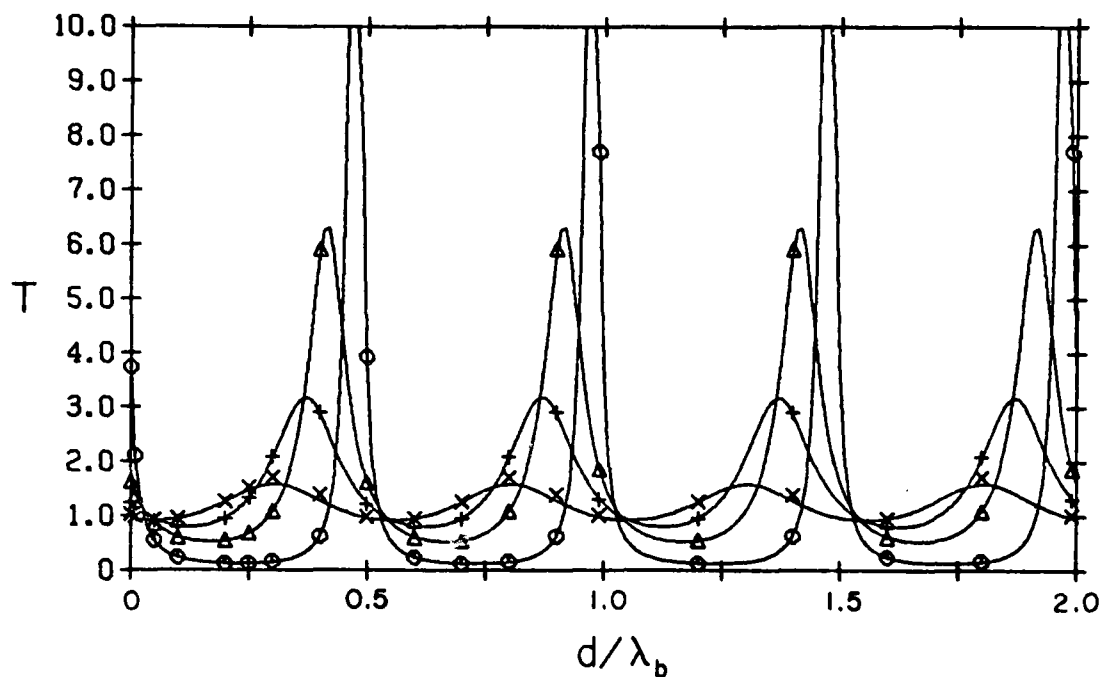
Fig. 6-5. Plots of transmission coefficient T vs. d/λ_b for cases of
Fig. 6-4. Width $w = 0.01\lambda_o$ (circles), $w = 0.05\lambda_o$ (triangles),
$w = 0.1\lambda_o$ (pluses), and $w = 0.2\lambda_o$ (crosses).

When the slot is loaded with different dielectrics, the
imaginary part of Y_{12} is most affected. This is shown in Fig. 6-6 for a
slot of width $w = 0.1\lambda_o$ for $\varepsilon_b = \varepsilon_o$, $5\varepsilon_o$, and $10\varepsilon_o$, where ε_o is the
permittivity of free space. As the material filling the slot becomes more
dense, the transmission resonances become narrower and occur closer to
multiples of $\lambda_b/2$. This is shown in Fig. 6-7 for the slot of Fig. 6-6,
the result being in agreement with (6-43). Note that the heights of the
peaks, however, are the same.

1.0
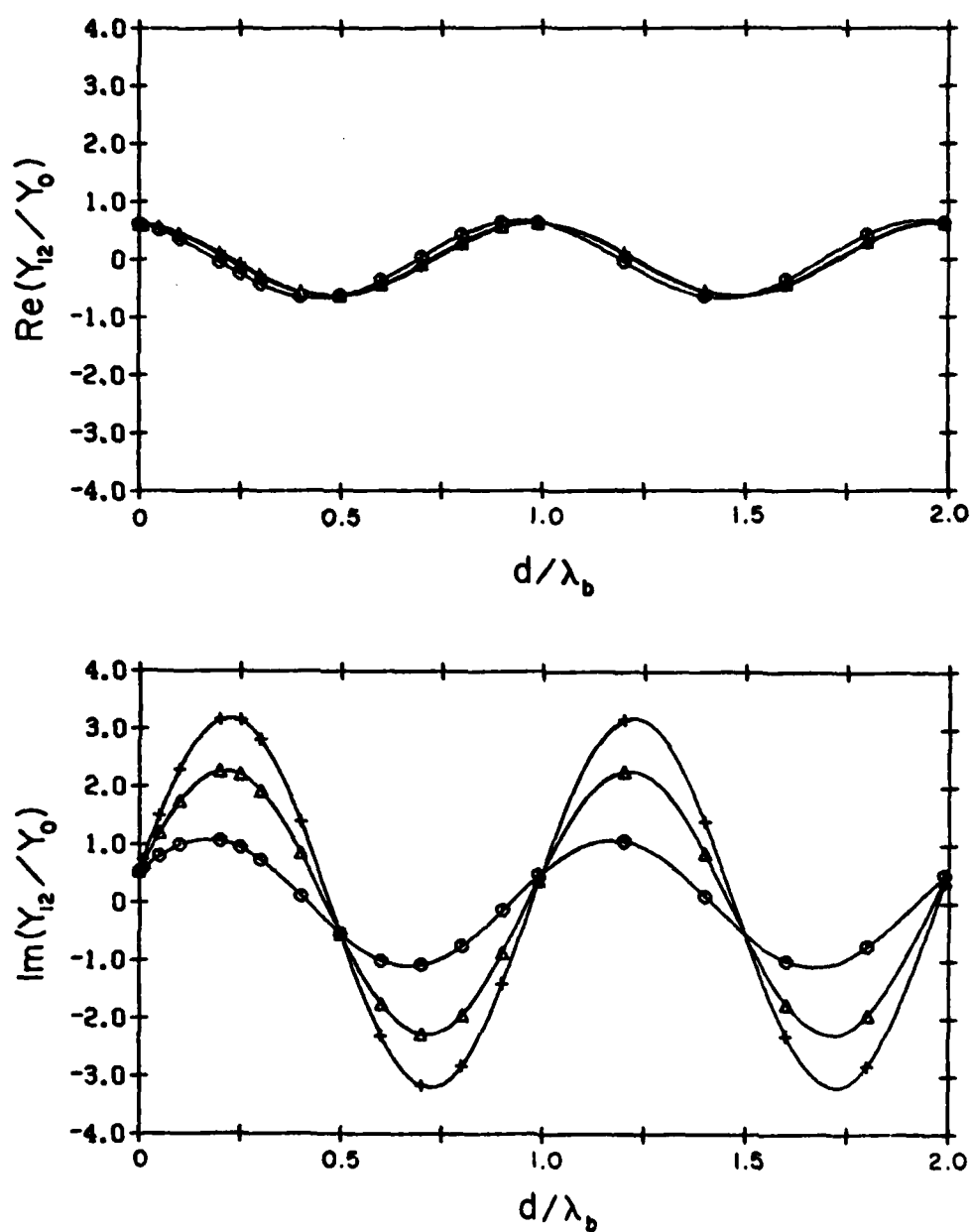
1.1

1.25 1.4 1.6

1.8

2.0

2.2

2.5
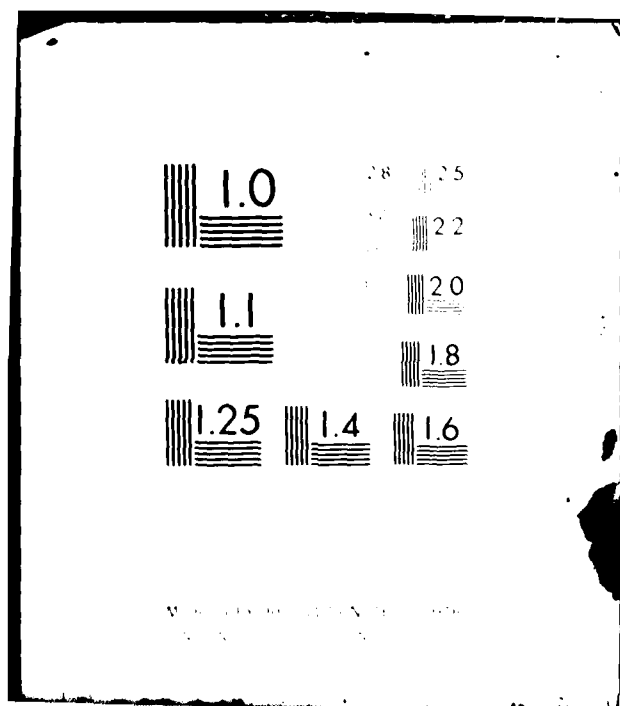
Fig. 6-6. Plots of $\text{Re}(Y_{12}/Y_o)$ and $\text{Im}(Y_{12}/Y_o)$ vs. d/λ_b for $w = 0.1\lambda_o$, $\varepsilon_a = \varepsilon_c = \varepsilon_o$, with various dielectrics filling the slot. Cases shown are $\varepsilon_b = \varepsilon_o$ (circles), $\varepsilon_b = 5\varepsilon_o$ (triangles), and $\varepsilon_b = 10\lambda_o$ (pluses).
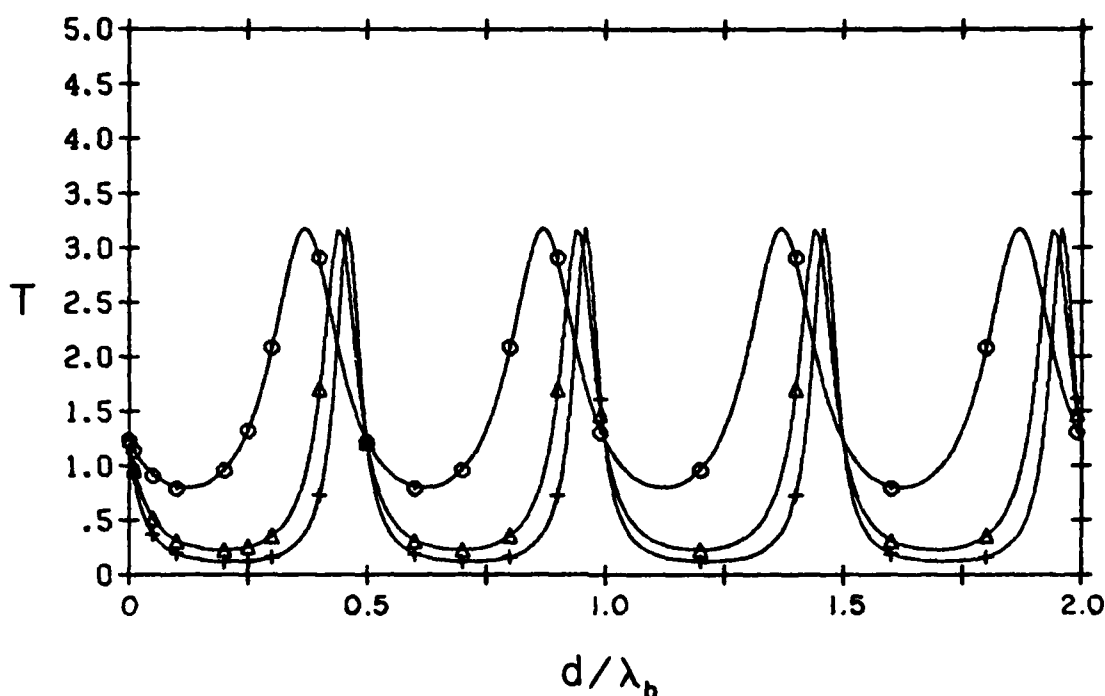
Fig. 6-7. Plots of transmission coefficient T vs. d/λ_b for the case- of Fig. 6-6. The dielectrics filling the slot are $\varepsilon_b = \varepsilon_o$ (circles), $\varepsilon_b = 5\varepsilon_o$ (triangles), and $\varepsilon_b = 10\varepsilon_o$ (pluses).

The effects of lossy material filling the slot are shown in Fig. 6-8. Here the slot width is $0.05\lambda_o$. The expected decay in the transmission coefficient peaks is seen to occur with increasing screen thickness d. Circles represent the lossless case $\varepsilon_b = \varepsilon_o$, triangles represent a dielectric $\varepsilon_b = (1-j0.01)\varepsilon_o$, that is, with Q factor = 100, and pluses represent a dielectric $\varepsilon_b = (1-j0.1)\varepsilon_o$, that is, with Q factor = 10. Note that loss does not affect the position of the resonances.

If the conductor has large but finite conductivity, the effect of the conductor loss is similar to that of dielectric loss. For small
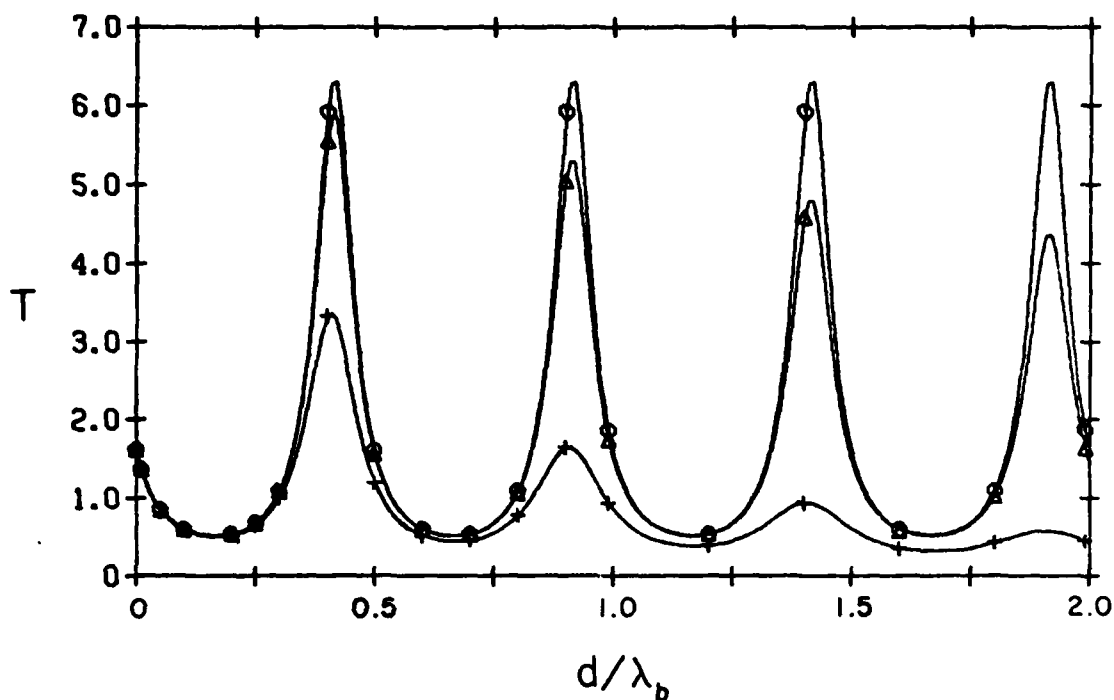
Fig. 6-8. Plots of transmission coefficient T vs. d/λ_b for $\varepsilon_a = \varepsilon_c = \varepsilon_o$, $w = 0.05\lambda_o$, for lossy dielectric in the slot. Cases shown are $\varepsilon_b = \varepsilon_o$ (circles), $\varepsilon_b = (1-j0.01)\varepsilon_o$ (triangles), and $\varepsilon_b = (1-j0.1)\varepsilon_o$ (pluses).

losses, either conductor or dielectric, the principal effect is to change the propagation constant of the parallel-plate transmission line mode from purely imaginary to complex with a s' '' nal part [11]. Equating the attenuation constant for a parallel-platc iveguide with perfectly conducting walls and lossy dielectric ($\varepsilon = \varepsilon' - j\varepsilon''$ and quality factor $Q = \varepsilon'/\varepsilon''$) to one with perfect dielectric and lossy conducting walls (surface resistance $R_s = \sqrt{\omega\mu/2\sigma_{cond}}$), we obtain

$$Q = \frac{k_b \eta_b w}{2R_s}$$

(6-49)

Now the lossy conductor case behaves similarly to the lossy dielectric case with Q factor given by (6-49). Note that Q is proportional to w, meaning that for very narrow slots the losses due to conducting walls become an important factor limiting the field penetration through slots in conductors of resonant thickness.

6-5 CONCLUSION

The validity of an equivalent circuit model for a narrow TE excited slot in a thick conducting screen has been investigated. The equivalent circuit formulas for transfer admittance and transmission coefficient give accurate results when compared to a higher order moment solution. When the material filling the slot is dense enough to allow more than one propagating mode, the equivalent circuit picture of Fig. 6-3 becomes more complicated. One must then consider a sequence of transmission lines connecting regions a and c in which the higher order modes couple to one another.

An interesting result of this investigation is that Tw, the transmission width of the slot, for a narrow slot at resonance, is independent of actual slot width. This phenomenon is analogous to the phenomenon of scattering by resonant scatters, or reception by resonant antennas. For example, a short dipole resonated by an inductor has a scattering cross section of $9/4\pi$ square wavelengths regardless of the actual size of the dipole [12]. A short dipole resonated by an inductor and used as a receiving antenna has an effective aperture of $3/8\pi$ square wavelengths regardless of the actual size of the dipole [13].

6-6   REFERENCES FOR CHAPTER VI

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Chapter VII

RESONANT BEHAVIOR OF A SMALL APERTURE

BACKED BY A CONDUCTING BODY

7-1 INTRODUCTION

The coupling of electromagnetic energy to wires or other conducting

objects through an aperture in a conducting wall is an important problem

in the theory of electromagnetic compatibility and interference. As an

approximation, some investigators have first solved the problem for the

tangential electric field in the aperture, and then taken this as a

secondary source for the field coupled to a wire [1]. This approach

neglects the effect of the wire on the electric field in the aperture,

which can be appreciable. Under certain conditions of resonance, the

power transmitted through an aperture can be orders of magnitude larger

when an object is near it than when no object is present.

Our approach is to first obtain the functional equations for the

problem using the equivalence principle (Sec. 3-5 of [2]), and then to

reduce these equations to matrix form via the method of moments [3]. The

various matrices are interpreted in terms of generalized network parame-

ters, such as voltages, currents, admittances, and impedances [3]. The

aperture admittance matrices of electrically small apertures are obtained

and discussed. An example of coupling through a capacitively loaded

aperture is given to illustrate the phenomenon of aperture resonance.

Perhaps the first accurate treatment of coupling through an aper-

ture to a wire was given by Butler and Umashankar [4]. Later work by

Butler considered the electrically small aperture and coupling to

objects other than wires [5]. He used Bethe-hole theory for the small
aperture but did not include a radiation term. He also did not look at
the resonance effects which we emphasize. Other work which treats
coupling to wires is that of Kajfez [6], Kajfez and Wilton [7], and Lee
and Yang [8]. These last three references deal with determining an
equivalent circuit for wires passing near small apertures. Again Bethe-
hole theory was used, radiation was not accounted for in the equivalent
circuit, and resonance effects were not considered.

7-2 FORMULATION OF THE PROBLEM

The general problem of coupling to a conducting body through an
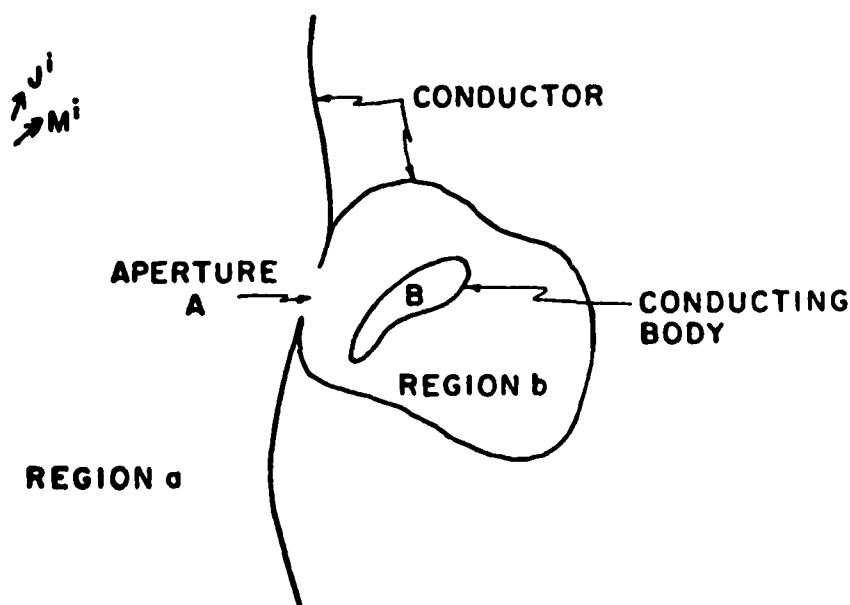aperture in a conducting wall is represented by Fig. 7-1. The wall

Fig. 7-1. Two-region aperture problem with impressed sources J^i, M^i
in region a and conducting object in region b.

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divides space into two regions, called region a and region b. The excitation is represented by impressed sources in region a, and the conducting body is in region b. The problem is primarily that of finding the tangential electric field in the aperture and the current on the conducting body, and secondarily that of finding the fields throughout space.

We use the equivalence principle (Section 1-6) to divide the problem into two parts, as shown in Fig. 7-2. The field in region a remains unchanged if the aperture is closed by a conductor and the equivalent surface magnetic current

$$\underline{M} = \underline{n} \times \underline{E} \tag{7-1}$$

is placed over the aperture region, where $\underline{E}$ is the electric field in the aperture of the original problem and $\underline{n}$ is the unit normal pointing outward. The original sources $\underline{J}^i$, $\underline{M}^i$ must be kept in region a. This equivalence is shown in Fig. 7-2a. The field in region b remains unchanged if the aperture is closed by a conductor and the equivalent surface magnetic current $-\underline{M}$ is placed over the aperture region. The current $\underline{J}$ on the conducting body must be kept in region b. This equivalence is shown in Fig. 7-2b.

The fact that $\underline{M}$ is used in Fig. 7-2a and $-\underline{M}$ is used in Fig. 7-2b ensures that the tangential electric field is equal on each side of the aperture region. We have two more boundary conditions to enforce: (a) the tangential magnetic field must be equal on each side of the aperture region, and (b) the tangential electric field must vanish on the surface of the conducting body. These two conditions give us two equations from which to calculate the unknown quantities $\underline{M}$ and $\underline{J}$.
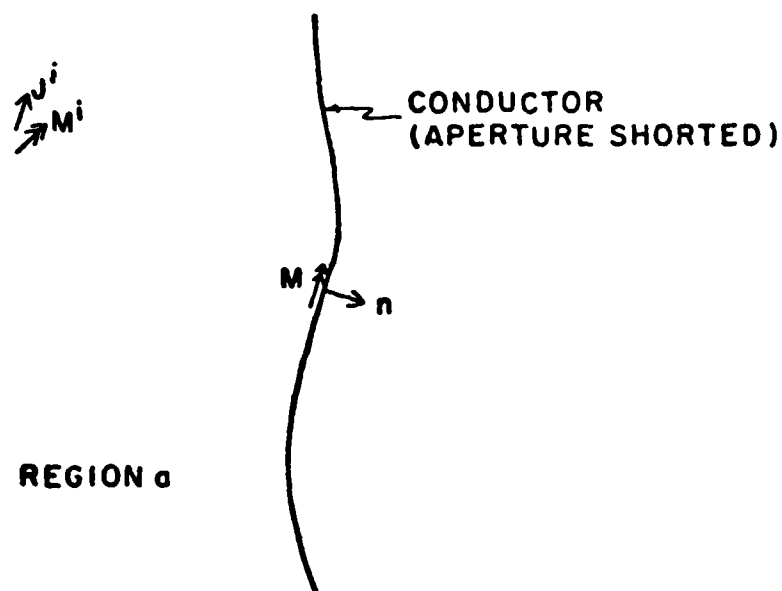
(a) EQUIVALENCE FOR REGION a
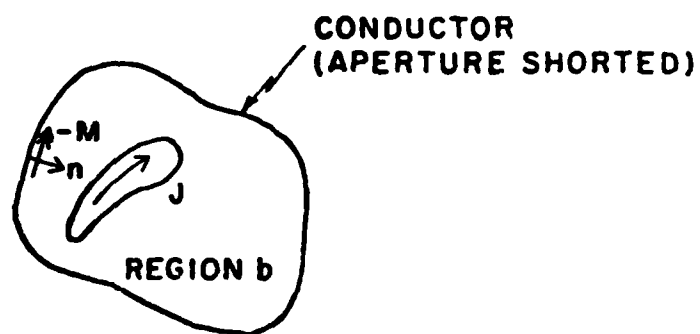
(b) EQUIVALENCE FOR REGION b

Fig. 7-2. Original problem divided into two equivalent parts.

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To express these boundary conditions in equation form, we intro-
duce the following notation: Let the electric and magnetic fields in
region a be denoted

$$\underline{E}^a = \underline{E}^a(\underline{M}) + \underline{E}^i \tag{7-2}$$

$$\underline{H}^a = \underline{H}^a(\underline{M}) + \underline{H}^i \tag{7-3}$$

where $\underline{E}^a(\underline{M})$, $\underline{H}^a(\underline{M})$ are the fields from $\underline{M}$ in Fig. 7-2a, and $\underline{E}^i$, $\underline{H}^i$ are
the fields from $\underline{J}^i$, $\underline{M}^i$ in Fig. 7-2a. Note that all fields are computed
with the aperture shorted. Similarly, let the electric and magnetic
fields in region b be denoted

$$\underline{E}^b = \underline{E}^b(-\underline{M}) + \underline{E}^b(\underline{J}) \tag{7-4}$$

$$\underline{H}^b = \underline{H}^b(-\underline{M}) + \underline{H}^b(\underline{J}) \tag{7-5}$$

where $\underline{E}^b(-\underline{M})$, $\underline{H}^b(-\underline{M})$ are the fields from $-\underline{M}$ in Fig. 7-2b, and $\underline{E}^b(\underline{J})$,
$\underline{H}^b(\underline{J})$ are the fields from $\underline{J}$ in Fig. 7-2b. Again, all fields are
computed with the aperture shorted.

To satisfy the boundary condition that the tangential component
of $\underline{H}$ must be continuous across the aperture, we equate tangential com-
ponents of (7-3) and (7-5), obtaining

$$-\underline{H}^a_t(\underline{M}) - \underline{H}^b_t(\underline{M}) + \underline{H}^b_t(\underline{J}) = + \underline{H}^i_t \quad \text{over A} \tag{7-6}$$

where A denotes the aperture region. The subscripts t denote the tan-
gential component over A, and we have used the linearity of the operator
to replace $\underline{H}^b_t(-\underline{M})$ by $-\underline{H}^b_t(\underline{M})$. To satisfy the boundary condition that the
tangential component of $\underline{E}$ must vanish on the surface of the conducting
body, we set the tangential component of (7-4) equal to zero, obtaining

$$\underline{E}_t^b(\underline{M}) - \underline{E}_t^b(\underline{J}) = 0 \qquad \text{over B} \tag{7-7}$$

where B denotes the conducting body surface. The subscripts t denote the tangential component over B, and we have used the linearity of the operator to replace $\underline{E}_t^b(-\underline{M})$ by $-\underline{E}_t^b(\underline{M})$. Equations (7-6) and (7-7) are vector equations for determining the unknowns $\underline{M}$ and $\underline{J}$ which exist over the surfaces A and B.

We next reduce (7-6) and (7-7) to matrix equations using the method of moments [3]. For this, we define a set of expansion functions $\{\underline{M}_n, \ n = 1, 2, \ldots, \ N_A\}$ and express the magnetic current over A as

$$\underline{M} = \sum_n V_n \underline{M}_n \tag{7-8}$$

where V_n are coefficients to be determined. We define a set of expansion functions $\{\underline{J}_n, \ n = 1, 2, \ldots, \ N_B\}$ and express the electric current over B as

$$\underline{J} = \sum_n I_n \underline{J}_n \tag{7-9}$$

where I_n are coefficients to be determined. We substitute (7-8) and (7-9) into (7-6) and (7-7), use the linearity of the operators, and obtain

$$-\sum_n V_n \underline{H}_t^a(\underline{M}_n) - \sum_n V_n \underline{H}_t^b(\underline{M}_n) + \sum_n I_n \underline{H}_t^b(\underline{J}_n) = \underline{H}_t^i \tag{7-10}$$

over A, and

$$\sum_n V_n \underline{E}_t^b(\underline{M}_n) - \sum_n I_n \underline{E}_t^b(\underline{J}_n) = 0 \tag{7-11}$$

over B. For A, we define a set of testing functions $\{\underline{M}_m, \ m = 1, 2, \ldots, \ N_A\}$ and a symmetric product

$$\langle F, \ G \rangle_A = \iint\limits_A \underline{F} \cdot \underline{G} \ ds \tag{7-12}$$

We take the symmetric product of (7-10) with each $\hat{\underline{M}}_m$, and use the linearity of the symmetric product to obtain

$$- \sum_n V_n <\hat{M}_m, H_t^a(M_n)>_A - \sum_n V_n <\hat{M}_m, H_t^b(M_n)>_A$$

$$\text{(7-13)}$$

$$+ \sum_n I_n <\hat{M}_m, H_t^b(J_n)>_A = <\hat{M}_m, H_t^i>_A$$

$m = 1, 2, \ldots, N_A$. For B, we define a set of testing functions $\{\hat{\underline{J}}_m, m = 1, 2, \ldots, N_B\}$ and a symmetric product

$$<F, G>_B = \iint_B \underline{F} \cdot \underline{G}\ ds \qquad \text{(7-14)}$$

We take the symmetric product of (7-11) with each $\hat{\underline{J}}_m$, and use the linearity of the symmetric product to obtain

$$\sum_n V_n <\hat{J}_m, E_t^b(M_n)>_B - \sum_n I_n <\hat{J}_m, E_t^b(J_n)>_B = 0 \qquad \text{(7-15)}$$

$m = 1, 2, \ldots, N_B$. Equations (7-13) and (7-15) are now a set of algebraic equations for determining the unknown coefficients V_n and I_n.

The above equations can be put into matrix notation as follows: Define an admittance matrix for region a as

$$[Y^a] = [<-\hat{M}_m, H_t^a(M_n)>_A]_{N_A \times N_A} \qquad \text{(7-16)}$$

and an admittance matrix for region b as

$$[Y^b] = [<-\hat{M}_m, H_t^b(M_n)>_A]_{N_A \times N_A} \qquad \text{(7-17)}$$

Note that these are exactly the same as defined in Chapter II for the

problem with the wire absent. Define coupling matrices

$$[T] = [<\hat{M}_m, H_t^b(J_n)>_A]_{N_A \times N_B} \tag{7-18}$$

and

$$[\hat{T}] = [<\hat{J}_m, E_t^b(M_n)>_B]_{N_B \times N_A} \tag{7-19}$$

Define an impedance matrix for the wire object as

$$[Z] = [<-\hat{J}_m, E_t^b(J_n)>_B]_{N_B \times N_B} \tag{7-20}$$

Note that this is calculated in the presence of a complete conducting boundary, that is, with the aperture shorted. Define a source vector

$$\vec{I}^i = [<\hat{M}_m, H_t^i>_A]_{N_A \times 1} \tag{7-21}$$

and coefficient vectors

$$\vec{V} = [V_n]_{N_A \times 1} \tag{7-22}$$

$$\vec{I} = [I_n]_{N_B \times 1} \tag{7-23}$$

Now the matrix equations equivalent to (7-13) and (7-15) are

$$[Y^a + Y^b]\vec{V} + [T]\vec{I} = \vec{I}^i \tag{7-24}$$

$$[\hat{T}]\ \vec{V} + [Z]\vec{I} = \vec{0} \tag{7-25}$$

The coefficient vectors $\vec{V}$ and $\vec{I}$ are obtained from the matrix solution to (7-24) and (7-25). If we want only $\vec{I}$, we can eliminate $\vec{V}$ from (7-25) by using (7-24). Similarly, if we want only $\vec{V}$, we can eliminate $\vec{I}$ from (7-24) by using (7-25).

For emphasis, we restate that $[Y^b]$ is calculated for the aperture as if the conducting body were not present, by methods outlined in Chapter II and carried out in detail for a rectangular aperture in a plane conducting wall in Chapter III. The matrix $[Z]$ is calculated for the conducting body as if the aperture were not present, that is, with the aperture short circuited. Conceptually, the only new matrices required are the interaction matrices, $[T]$ and $[\hat{T}]$. If a Galerkin solution is used, that is, if $\{M_n\} = \{\hat{M}_n\}$ and $\{J_n\} = \{\hat{J}_n\}$, it then follows from reciprocity (Sec. 3-8 of [2]) that $[\hat{T}] = - [\tilde{T}]$, where the tilde denotes transpose.

7-3 APERTURE ADMITTANCE MATRIX FOR SMALL HOLES

An aperture admittance matrix for electrically small apertures can be obtained from the quasi-static solution to the integral equation. The complete solution depends on both regions a and b, but the components of the aperture admittance matrix depend only on one region. The usual procedure is to solve for the aperture admittance of a canonical problem, and then use this solution for other problems which differ only slightly from the canonical problem. The basic canonical problem for aperture admittance is that of an aperture in an infinite plane conducting screen, excited by an incident plane wave. The electrostatic and magnetostatic treatment of this problem is known as Bethe-hole theory [10], [11].

The field produced by an aperture is defined to be the difference between that produced by the original sources in the presence of a shorted aperture and that produced when the aperture is present.

Except in the immediate vicinity of the aperture, the field of an electrically small aperture in a conducting screen is that of a magnetic dipole $\underline{p}_m$ tangential to the screen plus that of an electric dipole $\underline{p}_e$ normal to the screen, both radiating with the aperture shorted. Let the two half space regions on each side of the screen be denoted region a and region b, with sources possibly in both regions. Let $\underline{H}^{sca}$, $\underline{E}^{sca}$ and $\underline{H}^{scb}$, $\underline{E}^{scb}$ be the fields produced by these sources when the aperture is shorted. Then the difference field in region b is given by that from the dipoles

$$\underline{p}_m = - \bar{\alpha}_m \cdot (\underline{H}^{sca} - \underline{H}^{scb}) \tag{7-26}$$

$$\underline{p}_e = \epsilon \, \bar{\alpha}_e \cdot (\underline{E}^{sca} - \underline{E}^{scb}) \tag{7-27}$$

where $\bar{\alpha}_m$ and $\bar{\alpha}_e$ are the tensor magnetic and electric polarizabilities

$$\bar{\alpha}_m = \alpha_{m1}\underline{t}_1\underline{t}_1 + \alpha_{m2}\underline{t}_2\underline{t}_2 \tag{7-28}$$

$$\bar{\alpha}_e = \alpha_e \, \underline{n}\,\underline{n} \tag{7-29}$$

Here $\underline{t}_1$ and $\underline{t}_2$ are unit vectors tangential to the screen chosen to diagonalize $\bar{\alpha}_m$, and $\underline{n}$ is the unit normal to the screen pointing into region b. Analytical solutions for $\bar{\alpha}_m$ and $\bar{\alpha}_e$ exist for circular apertures [10] and elliptical apertures [12]. Methods for computing $\bar{\alpha}_m$ and $\bar{\alpha}_e$ for other shapes have been developed by De Meulenaere and Van Bladel [13], and of measuring them by Cohn [14], [15]. For an aperture with an axis of symmetry, $\underline{t}_1$ is in the direction of this axis and $\underline{t}_2$ is normal to it.

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The magnetic dipole moment $\underline{p}_m$ is that of a magnetic <u>pole</u> dipole and the electric dipole moment $\underline{p}_e$ is that of an electric <u>charge</u> dipole. For our purposes it is more convenient to deal with current elements, magnetic denoted by $K\underline{\ell}$ and electric denoted by $I\underline{\ell}$. These are related to $\underline{p}_m$ and $\underline{p}_e$ by

$$K\underline{\ell} = j\omega\mu \; \underline{p}_m \tag{7-30}$$

$$I\underline{\ell} = j\omega \; \underline{p}_e \tag{7-31}$$

Finally, let sources exist only in region a, and denote $\underline{E}^{sca}$, $\underline{H}^{sca}$ by $\underline{E}^i$, $\underline{H}^i$, as was done in section 7-2 or in Chapter II. Then (7-30) and (7-31), with substitutions from (7-26) and (7-27), become

$$K\underline{\ell} = - j\omega\mu \; \bar{\alpha}_m \cdot \underline{H}^i(0) \tag{7-32}$$

$$I\underline{\ell} = j\omega\epsilon \; \bar{\alpha}_e \cdot \underline{E}^i(0) \tag{7-33}$$

The notation $\underline{E}^i(0)$, $\underline{H}^i(0)$ denotes the value of $\underline{E}^i$ and $\underline{H}^i$ at the center 0 of the aperture. These impressed fields are assumed to be approximately constant over the aperture region.

We wish to translate the above results into those of aperture admittance, and to augment them to include the effects of radiation. Hence, for a small aperture, let the magnetic current be expressed by a three term expansion

$$\underline{M} = V_1\underline{M}_1 + V_2\underline{M}_2 + V_3\underline{M}_3 \tag{7-34}$$

where $\underline{M}_1$ is the quasi-static current which produces the effect of a unit magnetic dipole $K\ell = 1$ in the $\underline{t}_1$ direction, $\underline{M}_2$ is the quasi-static

current which produces the effect of a unit magnetic dipole $K\ell = 1$ in
the $\underline{t}_2$ direction, and $\underline{M}_3$ is the quasi-static current which produces the
effect of an electric dipole $I\ell = j$ in the $\underline{n}$ direction. (A real mag-
netic current gives rise to an imaginary $I\ell$ [2, p. 135].) We will not
need explicit formulas for $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$, but they can be obtained from
the Bethe-hole quasi-static solution.

The dipoles in (7-32) and (7-33) radiate in region b. In Fig. 7-2b,
$-\underline{M}$ radiates in region b. The dipole effects of $-\underline{M}$ will be equal to the
results (7-32) and (7-33) of Bethe-hole theory if the V's in (7-34) are
given by

$$V_1 = j\omega\mu\alpha_{m1}H_1^i(0)$$

$$V_2 = j\omega\mu\alpha_{m2}H_2^i(0) \tag{7-35}$$

$$V_3 = - \omega\varepsilon\alpha_e E_3^i(0)$$

Here, the subscripts 1, 2, and 3 denote the $\underline{t}_1$, $\underline{t}_2$, and $\underline{n}$ directions,
respectively. For our canonical problem involving the infinite plane
conducting screen, $\vec{I} = \vec{0}$ in (7-24) so that the method of moments solution
for $\vec{V}$ reduces to

$$\vec{V} = [Y^a + Y^b]^{-1} \vec{I}^i \tag{7-36}$$

For testing functions we use $\hat{\underline{M}}_m = \underline{M}_m$, n = 1,2,3 (Galerkin's method). We
then find

$$\vec{I}^i = \begin{bmatrix} H_1^i(0) \\ H_2^i(0) \\ -jE_3^i(0) \end{bmatrix} \tag{7-37}$$

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Now for the admittance solution (7-36) to give the same result as Bethe-hole theory, we must have

$$[Y^a + Y^b] = \begin{bmatrix} \dfrac{1}{j\omega\mu\alpha_{m1}} & 0 & 0 \\[3mm] 0 & \dfrac{1}{j\omega\mu\alpha_{m2}} & 0 \\[3mm] 0 & 0 & \dfrac{-1}{j\omega\epsilon\alpha_e} \end{bmatrix} \qquad (7\text{-}38)$$

The solution (7-36) gives coefficients V_n which, according to (7-34), give the negative of the dipoles specified by (7-32) and (7-33).

The aperture admittance matrix obtained from Bethe-hole theory neglects radiation. In other words, it evaluates only the first term of a frequency expansion for $[Y^a + Y^b]$. When there is an object near the aperture, as in our original problem, the first (susceptive) term in $[Y^a + Y^b]$ can be cancelled by the interaction between the object and the aperture, resulting in the incorrect prediction of infinite coefficients V_n, or infinite power transmitted through the aperture. This defect in the Bethe-hole theory can be corrected by evaluating the second (conductive) term of a frequency expansion for $[Y^a + Y^b]$. Fortunately, this additional term is easily obtained from the radiation field of the dipoles.

Note that, for $\underline{M}_n$ real, we have $\mathrm{Re}(Y^a_{nn} + Y^b_{nn})$ equal to 4 times the power radiated by $\underline{M}_n$ in free space. The factor of 4 arises because $\underline{H}^a_t(\underline{M}_n)$ in (7-16) is twice the magnetic field due to $\underline{M}_n$ radiating in free space and because $[Y^a + Y^b] = 2[Y^a]$. The field distant from $\underline{M}_n$ is assumed to be a dipole field, hence we can use the results for a dipole. Using duality (Sec. 3-2 of [2]) and the formula for the power radiated by an

electric dipole (Eq. 2-116 of [2]), we have for a magnetic dipole

$$P = \frac{2\pi}{3\eta} \left| \frac{K\ell}{\lambda} \right|^2 \tag{7-39}$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of space and λ is the wavelength. Since $\underline{M}_1$ and $\underline{M}_2$ each produce the effect of $K\ell = 1$, we have

$$\text{Re}[Y^a + Y^b]_{11} = \text{Re}[Y^a + Y^b]_{22} = 4P = \frac{8\pi}{3\eta\lambda^2} \tag{7-40}$$

By similar reasoning, for the electric dipole $I\ell = j$ radiating on the surface of an infinite conducting plane we obtain a term

$$\text{Re}[Y^a + Y^b]_{33} = \frac{8\pi\eta}{3\lambda^2} \tag{7-41}$$

These terms must be added to the imaginary terms in (7-38) to give the aperture admittances corrected to account for radiation. The off-diagonal elements in (7-38) remain zero because there is no cross-power between any pair of the quasi-static currents $\underline{M}_1$, $\underline{M}_2$, and $\underline{M}_3$.

For a half-space region alone the matrix elements are one-half those evaluated above, or

$$[Y^{hs}] = \begin{bmatrix} Y^{hs}_{11} & 0 & 0 \\ 0 & Y^{hs}_{22} & 0 \\ 0 & 0 & Y^{hs}_{33} \end{bmatrix} \tag{7-42}$$

where

$$Y^{hs}_{11} = \frac{1}{2j\omega\mu\alpha_{m1}} + \frac{4\pi}{3\eta\lambda^2}$$

$$Y^{hs}_{22} = \frac{1}{2j\omega\mu\alpha_{m2}} + \frac{4\pi}{3\eta\lambda^2} \tag{7-43}$$

$$Y^{hs}_{33} = \frac{-1}{2j\omega\epsilon\alpha_e} + \frac{4\pi\eta}{3\lambda^2}$$

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Of course, equations (7-43) are strictly valid only for a half space
region, but they are often a good approximation for other regions.

7-4 TRANSMITTED POWER

The complex power through the aperture is basically given by

$$P_A = \iint\limits_{A} \underline{E} \times \underline{H}^* \cdot \underline{n} \; ds$$

$$= \iint\limits_{A} \underline{M} \cdot \underline{H}^* \; ds \tag{7-44}$$

where $\underline{M}$ is the equivalent magnetic current defined by (7-1), and $\underline{H}^*$ is
the conjugate of the magnetic field in the aperture. Setting $\underline{H} = \underline{H}^b$,
and using (7-5), we have

$$P_A = \iint\limits_{A} \underline{M} \cdot [-\underline{H}^b(\underline{M}) + \underline{H}^b(\underline{J})]^* \; ds$$

$$= -\iint\limits_{A} \underline{M} \cdot \underline{H}^{b*}(\underline{M}) ds + \iint\limits_{A} \underline{M} \cdot \underline{H}^{b*}(\underline{J}) \; ds \tag{7-45}$$

Substituting for $\underline{M}$ from (7-8) and for $\underline{J}$ from (7-9), we reduce this to

$$P_A = - \sum_m \sum_n V_m V_n^* \iint\limits_{A} \underline{M}_m \cdot \underline{H}^{b*}(\underline{M}_n) \; ds$$

$$+ \sum_m \sum_n V_m I_n^* \iint\limits_{A} \underline{M}_m \cdot \underline{H}^{b*}(\underline{J}_n) \; ds \tag{7-46}$$

If a Galerkin solution ($\hat{\underline{M}}_n = \underline{M}_n$ and $\hat{\underline{J}}_n = \underline{J}_n$) is used, and if the expansion

functions are real, then the negatives of the first integrals in (7-46) are Y_{mn}^{b*} as defined by (7-17), and the second integrals in (7-46) are T_{mn}^{*} as defined by (7-18). Hence,

$$P_A = \sum_m \sum_n V_m V_n^* Y_{mn}^{b*} + \sum_m \sum_n V_m I_n^* T_{mn}^*$$ (7-47)

In matrix form this becomes

$$P_A = \tilde{V}[Y^b]*\vec{V}* + \tilde{V}[T]*\vec{I}*$$ (7-48)

where $\vec{V}$ is defined by (7-22), $\vec{I}$ by (7-23), and the tilde denotes transpose.

The last term in (7-48) can be put into another form by using (7-25) to express $\vec{I}$ in terms of $\vec{V}$. Since $[\hat{T}] = - [\tilde{T}]$, we have

$$\vec{I} = [Z^{-1}\tilde{T}]\vec{V}$$ (7-49)

Substituting this into (7-48), and combining the two matrix products, we have

$$P_A = \tilde{V}[Y^b + TZ^{-1}\tilde{T}]* \vec{V}*$$ (7-50)

This suggests that we can define the effective aperture admittance into region b to be

$$[Y_{eff}^b] = [Y^b + TZ^{-1}\tilde{T}]$$ (7-51)

Then (7-50) becomes the usual formula for power transmitted into an N-port network. Note that the effective aperture admittance is that for region b with the conducting body present. In this case we are viewing the conducting body as part of the definition of region b, that is, part of the environment into which the aperture radiates.

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We can see more clearly what can happen if the magnetic current $\underline{M}$ in the aperture and the electric current $\underline{J}$ on the body are expressed in terms of a single expansion function each. These two quantities are then given by

$$\underline{M} = V\underline{M}_1 \qquad \text{and} \qquad \underline{J} = I\underline{J}_1 \qquad\qquad (7\text{-}52)$$

where only the complex amplitudes V and I are unknown. The matrix equations (7-24) and (7-25) are now reduced to the scalar equations

$$YV + TI = H_t^{sca} \qquad\qquad (7\text{-}53)$$

$$-TV + ZI = 0 \qquad\qquad (7\text{-}54)$$

where $Y = Y^a + Y^b$, T, and Z are given by (7-16, 17, 18, and 20) with the subscripts m and n each being 1 only. Expressing I in terms of V from (7-54), and substituting into (7-53), we have

$$(Y + \frac{T^2}{Z})V = H_t^{sca} \qquad\qquad (7\text{-}55)$$

For electrically small apertures, normally $|\text{Im}(Y)| \gg \text{Re}(Y)$. We define aperture-body resonance to be the case for which

$$\text{Im}(T^2/Z) = - \text{Im}(Y) \qquad\qquad (7\text{-}56)$$

T^2 is real positive if the body and aperture are electrically small and close to each other. For this case, at resonance (7-55) reduces to

$$(G + \frac{T^2 R}{|Z|^2}) \, V_r = H_t^{sca} \qquad\qquad (7\text{-}57)$$

where the subscript r on V_r denotes "at aperture-body resonance," $G = \text{Re}(Y)$, and $R = \text{Re}(Z)$.

In the next section we show by example that the power transmitted by an electrically small aperture backed by a conducting body can be much larger when the body is present than when the body is absent. As a limiting case, the second term in parenthesis in (7-55) may be small compared to the first term. When the aperture is far from resonance, we then have

$$V \approx \frac{H_t^{sca}}{Im(Y)} \qquad (7-58)$$

When aperture-body resonance occurs, we have

$$V_r \approx \frac{H_t^{sca}}{Re(Y)} \qquad (7-59)$$

We define the aperture Q as

$$Q = \frac{|Im(Y)|}{Re(Y)} \qquad (7-60)$$

Then, since $\underline{M} = V\underline{M}_1$, we see that $\underline{E}_t$ in the aperture can be increased at most by the factor Q at aperture-body resonance. The power transmitted by the aperture, (7-50), depends on $|V|^2$, and hence may be increased at most by Q^2. As a word of caution, note that Q depends on $\underline{M}_1$ as well as on the aperture size and shape.

7-5 AN EXAMPLE

As an example, consider a small rectangular aperture in a plane conducting screen with a capacitor placed across its midpoint, as shown in Fig. 7-3a. Take the excitation to be a plane wave normally incident on the screen with $\underline{E}^i$ perpendicular to the longer axis of the slot. By Babinet's principle (Sec. 7-12 of [2]), this problem is dual to that of
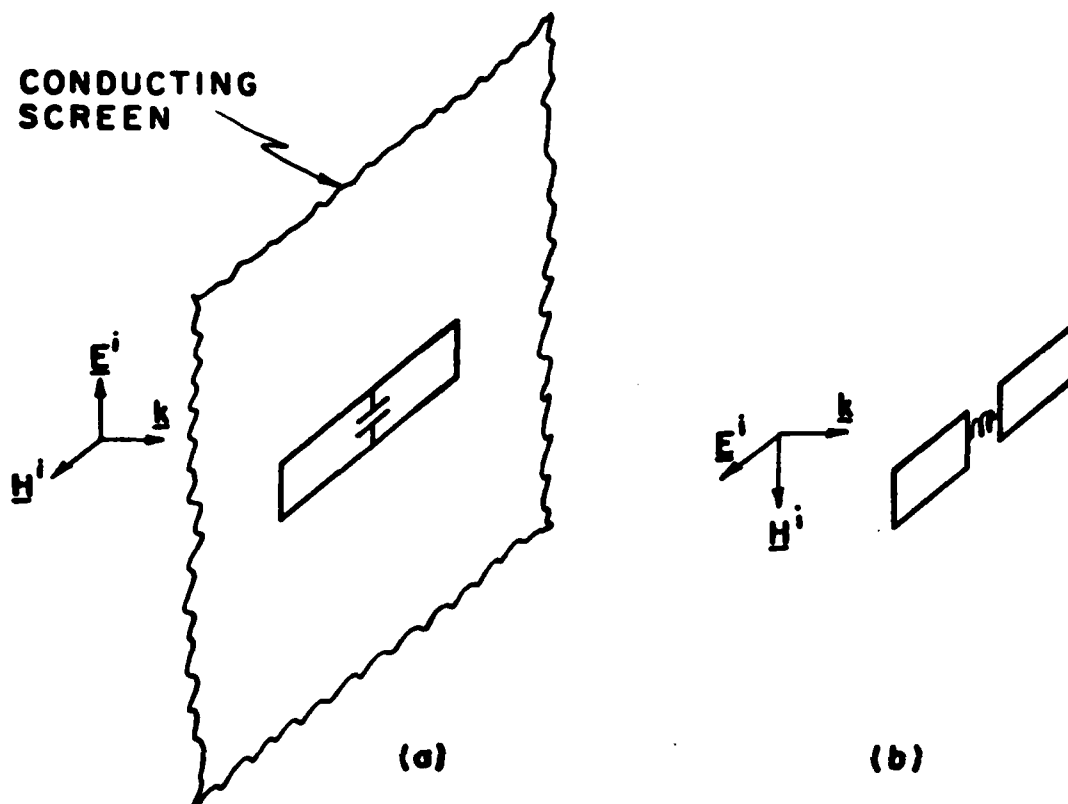
Fig. 7-3.   (a) A capacitively loaded aperture and (b) the
complementary inductively loaded scatterer.

electromagnetic scattering from a conducting rectangular dipole, with

an inductor in series at its midpoint, as shown in Fig. 7-3b. It is

known that, for a loss-free resonated dipole, the back scattering cross

section is

$$\sigma = \frac{9\lambda^2}{4\pi}$$

(7-61)

regardless of the size or shape of the dipole [16]. The result (7-61)

can, however, be greatly decreased by conductor losses for small dipoles.

We now show that a similar result holds for the transmission cross section

of a small resonated aperture.

For a plane wave normally incident on an aperture, we define the transmission coefficient T of the aperture to be the ratio of the power transmitted through the aperture to the power incident on it, that is,

$$T = \frac{Re(P_A)}{|H_o|^2 \eta A} \qquad (7\text{-}62)$$

Here A is the area of the aperture and $H_o = H_t^{sca}/2$ is the incident part of H^i. For the problem of Fig. 7-3, the current on the "conducting body," that is, the capacitor, is in phase quadrature with the voltage across it. Therefore the real part of the second term of (7-50) must be zero and, for scalar V and Y^b, it reduces to

$$Re(P_A) = |V|^2 G^b \qquad (7\text{-}63)$$

The conductance G^b is $Re(Y^b)$ which, from (7-43), is seen to be

$$G^b = \frac{4\pi}{3\eta\lambda^2} \qquad (7\text{-}64)$$

At resonance, $V = V_r$ is given by (7-57) with the second term zero, or

$$V_r = \frac{H_t^{sca}}{G} = \frac{H_t^{sca}}{2G^b} \qquad (7\text{-}65)$$

The second equality in (7-65) is due to the fact that G is the conductance seen by the aperture opening into two half spaces in parallel, while G^b is the conductance seen by the aperture looking into the single half space region b. Using (7-63) and (7-65) in (7-62), we have

$$T = \frac{|H_t^{sca}|^2}{|H_o|^2 \eta A \, 4 \, G^b} \qquad (7\text{-}66)$$

Finally, using (7-64) and the fact that $H_t^{sca} = 2H_o$, we obtain

$$T = \frac{1}{\eta A G^b} = \frac{3\lambda^2}{4\pi A} \tag{7-67}$$

Hence, the transmission coefficient of an electrically small aperture resonated by a capacitor is independent of the shape of the aperture. The transmission area of the aperture is defined as the area for which the incident wave contains the same power as transmitted by the aperture, that is

$$T\,A = \frac{3\lambda^2}{4\pi} \tag{7-68}$$

Hence, the transmission area of a small aperture resonated by a capacitor is $3\lambda^2/4\pi$ independent of the size and shape of the aperture. Of course, conductor losses may greatly decrease (7-68) for very small apertures.

7-6 DISCUSSION

This chapter illustrates how a conducting object situated near a small aperture in a conducting screen can resonate the aperture, thereby increasing the power transmitted by that aperture over what it would transmit with no object present. The magnitude of the increase can be of the order of Q^2, where Q is the quality factor of the aperture current being resonated. With no conductor losses Q becomes very large for electrically small holes. For example, the magnetic dipole of a circular aperture of radius R has a Q of 114 when $R = \lambda/20$ and a Q of 14,200 when $R = \lambda/100$. These values of Q were obtained from (7-60), (7-43), and, as given in [5], $\alpha_{m1} = (4/3)R^3$. When losses are included, we can expect Q to be limited to the order of a few hundred at radio frequencies.

The susceptance of the magnetic dipole mode for a small aperture
is inductive, and requires the coupling of a capacitive susceptance from
the backing object for resonance. In the example chosen, we obtained
this required susceptance by capacitively loading the slot. We used a
lumped load, but longer wires can produce this capacitive susceptance
without lumped loads. For example, a straight wire of length ℓ slightly
less than a half wavelength (a wire dipole) produces a capacitive sus-
ceptance. If region b is a cavity, it will reflect a capacitive sus-
ceptance at some frequencies and an inductive susceptance at other fre-
quencies. The reason we chose a lumped capacitive load was to show that
aperture-conductor resonance is possible even if all dimensions are
electrically small.

Another point that we wish to emphasize is that Bethe-hole theory
should assume that the form of the aperture field remains almost the same
as in the canonical problem of a small aperture in a plane screen, but
not its amplitude and phase. If this is not recognized, order of magni-
tude errors may result when the aperture interacts with other objects.
Also, the concept of small-aperture polarizability should be generalized
to that of aperture admittance by including a radiation conductance term.
If this is not done, infinities in the aperture power transmitted may occur.

The result that the transmission cross section of a small aperture
resonated by a capacitive load is independent of the size and shape of the
aperture is not unexpected, since similar results have been obtained in
other transmission problems. For example, the transmission width of a
narrow infinitely long slot in a thick conductor at resonance is λ/π,

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regardless of the actual slot width, as shown in Chapter VI. Of course,
this is for an ideal loss-free problem. Conductor losses can significantly
decrease the aperture transmission cross section for apertures in actual
conductors.

The simplest way to resonate a small aperture is to connect a
capacitor directly across the center of the aperture, as was done in the
example. This is an analogue to the problem of connecting an inductor
in series with a short dipole scatterer. When the aperture is in a shield,
we obviously do not want to resonate the aperture. However, when a broad
spectrum of frequencies is present, such as in an electromagnetic impulse,
an aperture-body system will possibly resonate at some frequencies of
interest. This possibility should be taken into account in the engineering
analysis of such a system.

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